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**Study 1:**


**Study 2:**

A General Family of Morphed Nonlinear Phase Oscillators with Arbitrary Limit Cycle Shape

Mostafa Ajallooeian\textsuperscript{a}, Jesse van den Kieboom\textsuperscript{a}, Albert Mukovskiy\textsuperscript{b}, Martin A. Giese\textsuperscript{b}, Auke J. Ijspeert\textsuperscript{a}

\textsuperscript{a}Biorobotics Laboratory, School of Engineering, École Polytechnique Fédérale de Lausanne (EPFL), Switzerland
\textsuperscript{b}Section for Computational Sensomotorics, Department of Cognitive Neurology, Hertie Institute for Clinical Brain Research & Center for Integrative Neuroscience, University Clinic Tübingen, Germany

Abstract

We present a general family of nonlinear phase oscillators which can exhibit arbitrary limit cycle shape and infinitely large basins of attraction. This general family is the superset of familiar control methods like PD-control over a periodic reference, and rhythmic Dynamical Movement Primitives. The general methodology is based on morphing the limit cycle of an existing phase oscillator with phase-based scaling functions to obtain a desired limit cycle behavior. The introduced methodology can be represented as first, second, or \( n \)-th order dynamical systems. The elegance of the formulation provides the possibility to define explicit arbitrary convergence behavior for simple cases. We analyze the stability properties of the methodology with the Poincaré-Bendixson theorem and the Contraction Theory, when possible, and use numerical simulations to show the properties of some oscillators that are a subset of this general family.

Keywords: nonlinear phase oscillator, arbitrary limit cycle shape, arbitrary convergence behavior, Poincaré-Bendixson theorem, Contraction Theory.

1. Introduction

Nonlinear oscillators have been widely used in different fields ranging from abstract to applied sciences and engineering [1]. Their capability of entrainment, synchronization and smooth modulation of their output signal makes them appropriate tools for modeling natural phenomena and for control [1–4].

One challenge in the field of nonlinear oscillators is how to design nonlinear oscillators with desired limit cycle shapes. Different methodologies attempt to solve this problem, including recurrent neural networks [5–12], adaptive frequency oscillators [13, 14], etc. However the solution to this problem still remains an open issue [4]. In this paper we propose a methodology which results in a general family of nonlinear phase oscillators exhibiting arbitrary limit cycle shapes.

Nonlinear oscillators with arbitrary limit cycle shape can be built with recurrent neural networks (RNN) [5–12]. However, it is not trivial to calculate the connection weights, analyze the asymptotic stability of the resulting network, and predict the behavior of it [8]. Nevertheless, there are reservoir computing approaches to RNN [15] which provide ways to create nonlinear oscillators without complicated training. Wyffels et al. [16] presented a randomly connected RNN with a linear readout which can encode arbitrary limit cycles, and used ridge regression to find the readout weights, yet the problem of asymptotic stability facing input or internal state perturbations remains unsolved (the basin of attraction is local and not global).

Some researchers have tried to create nonlinear oscillators with an arbitrary limit cycle shape by using fitting tools on data-driven generated vector fields. Okada et al. [17] presented a method to represent the desired trajectory as a limit cycle and define a corresponding vector field in the vicinity of each data point directed towards the limit cycle. They use a polynomial approximation of the vector field to create the dynamical system encoding the desired limit cycle. Similarly, Ajallooeian et al. [18] use the desired trajectory as the limit cycle and define margins of attraction based on the Poincaré-Bendixson theorem. They map the desired periodic trajectory on the limit cycle of a stable oscillator (and vice versa) to strengthen the stability properties, and employ feedforward neural networks to create the

maps. Both [17] and [18] create dynamical systems which are trained to be empirically (asymptotically) stable in the margins defined around the limit cycle, but not necessarily outside of these margins.

Righetti et al. [13] presented Adaptive Frequency Oscillators and used a pool of them to create nonlinear oscillators with an arbitrary limit cycle shape. They utilize a dynamic estimation of the phases of Fourier harmonics constructed from a pool of adaptive Hopf oscillators [19], and build the desired dynamics. The output of their model is the weighted sum of local oscillators, which makes the integration of the proprioceptive feedback difficult (also known as the credit assignment problem [20]). The final output shape of such system is limited, unless an infinite number of oscillators is used. Also a modified version of this approach was presented in [21] to use a single oscillator and an adaptive Fourier series.

One other interesting approach to create custom nonlinear oscillators is to use a virtual linear spring and periodically force it such that the desired output is generated. Ijspeert et al. [22–24] presented rhythmic Dynamical Movement Primitives (DMP) which implements this idea. The presented model is asymptotically stable and the passive damping diminish the state disturbances. Rhythmic DMP gives a nice formulation of a phase oscillator with an arbitrary limit cycle shape, and was later coupled with adaptive frequency oscillators [14]. Here in this paper we give a general representation which includes rhythmic DMP as an example realization.

We take inspiration from the works mentioned above, especially from our previously collaborated works [18] and [22–24], and present a general way to convert an existing phase oscillator to any desired nonlinear phase oscillator with well defined, controllable properties. We believe that our contribution is three fold: 1) We present a general and systematic way to design nonlinear phase oscillators. As a consequence, we obtain not a single, but a family of nonlinear phase oscillators exhibiting desired arbitrary limit cycle shapes. This gives researchers the ability to easily design custom nonlinear phase oscillators. 2) The general methodology we present here can be taken as a unifying view on well known techniques including PD control and rhythmic Dynamical Movement Primitives. This enables the comparison of these methods under a same conceptual framework. 3) The mathematical formulation gives room to introduce a simple nonlinear oscillator which can have a custom convergence behavior, independent of the other properties like arbitrary limit cycle shape. This is a useful property providing the possibility to define the way to converge to the limit cycle, which could be useful for initiation in many cyclic motion control problems.

We first discuss the general and basic methodology to create first order 1D nonlinear oscillators with arbitrary limit cycle shape by modifying existing phase oscillators. We then take a step by step procedure to introduce different variations of the methodology. This includes rewriting the methodology as second order dynamical systems, and then $n$-th order ones. We also extend the introduced oscillators to multi-dimensions by means of phase coupling, and introduce how to obtain custom convergence behavior in simple cases. The stability analyses for all these variations are included.

The rest of this document is organized as follows: The design methodology is explained in section 2. Example realizations of the introduced methodology is given in section 3. Extension to multi-dimensions is explained in section 4. We then analyze the stability conditions in section 5. Section 6 details how arbitrary convergence behavior can be obtained. Section 7 discusses building nonlinear phase oscillators from data. We summarize and discuss our work in section 8.

2. Methodology

There are many nonlinear oscillators with different characteristics, including Hopf oscillator [1], van der Pol oscillator [1], Fitzhugh-Nagumo oscillator [25], and many more, and each of these oscillators exhibit a different limit cycle shape. However, it is not trivial to design an oscillator with a desired arbitrary limit cycle shape. We propose a systematic way to design phase oscillators (oscillators with an explicit phase variable, such as a Hopf oscillator expressed in polar coordinates, for instance) with arbitrary limit cycle shape.

One can take a simple phase oscillator, and try to morph this oscillator’s limit cycle to obtain a desired one. So a mapping which takes the states of this oscillator and modifies them such that the outcome has the desired property is needed. This mapping can be arbitrarily complex, but in case of phase oscillators, we show that scaling the radial state depending on the phase value is sufficient to modify the limit cycle shape.

The main idea of the introduced methodology is based on shaping a simple low-dimensional oscillator’s limit cycle to obtain a desired limit cycle behavior. Our methodology consists of an oscillator called the base oscillator (in
Figure 1: Example of a mapping between the desired oscillator (left) and the base oscillator (right). The main idea is to select an existing structurally stable oscillator, like the Hopf oscillator (right), and find a mapping \( M \) that shapes the limit cycle of this oscillator to a desired one (left). A phase-based scaling function \( f(\theta) \) can implement \( M \) to map points from \( \mathbb{B} \) to \( \mathbb{S} \) and vice versa. For example, for the two points shown, the scaling function is \( f(0.9) = \frac{6}{7} \).

phase plane \( \mathbb{B} \), real plane \( \mathbb{R}^2 \), a desired oscillator (in phase plane \( \mathbb{S} \), real plane \( \mathbb{R}^2 \)) and a mapping between them \( (M) \). We aim to find invertible \( M \) such that any path in \( \mathbb{B} \) is mapped to a unique path in \( \mathbb{S} \) and vice versa. Consequently, with the correct choice of \( M \), the limit cycle of the base oscillator in \( \mathbb{B} \) will be mapped to the desired one in \( \mathbb{S} \).

Let us assume that the desired limit cycle is defined as function \( \Xi_B : \theta_B \rightarrow r_B \), with \( \theta_B \) and \( r_B \) respectively being the angle and radius in polar coordinates. Also assume a base oscillator for which its limit cycle can be defined in polar coordinates as function \( \Xi_B : \theta_B \rightarrow r_B \). Now if the oscillator in \( \mathbb{S} \) has a phase always equal to the base oscillator’s phase, i.e. \( \theta_B = \theta_S = \theta \), then the radius of the limit cycle of the oscillator in \( \mathbb{S} \) can be defined with a phase-based scaling (also see Figure 1):

\[
r_S = f(\theta)r_B
\]  

(1)

where \( f \) is a scaling function that scales based on the phase value. So a state \( \{\theta, r_S\} \) in \( \mathbb{S} \) corresponds to a state \( \{\theta, \frac{r_S}{f(\theta)}\} \) in \( \mathbb{B} \), and vice versa.

The mapping from space \( \mathbb{B} \) to \( \mathbb{S} \) can be written as \( M(\theta, r) = (\theta, rf(\theta)) \), and calculating the Jacobian determinant of \( M \) gives \( |J| = f(\theta) \). So if \( \forall \theta : f(\theta) \neq 0 \) and \( f \) is \( C^1 \) differentiable, then \( M \) defines a \( C^1 \)-diffeomorphism \([1]\) between \( \mathbb{B} \) and \( \mathbb{S} \). This means that each state in \( \mathbb{S} \) corresponds to a unique state in \( \mathbb{B} \) and vice versa. Having the diffeomorphism property, an intuitive way to form the desired limit cycle system in \( \mathbb{S} \) is\(^1\):

1. Take the current state \( \{\theta(t), r_S(t)\} \) in \( \mathbb{S} \)
2. Map this state to \( \mathbb{B} \) using \( r_B(t) = \frac{r_S(t)}{f(\theta(t))} \)
3. Use the dynamical system of the base oscillator in \( \mathbb{B} \) to calculate \( r_B(t + \Delta t) \) and \( \theta(t + \Delta t) \)
4. Map \( r_B(t + \Delta t) \) back to \( \mathbb{S} \) by \( r_S(t + \Delta t) = f(\theta(t + \Delta t)r_B(t + \Delta t) \)
5. Do \( t \leftarrow t + \Delta t \) and continue from 1.

With the above algorithm we can mathematically write the process of the limit cycle generation as an iterative map, i.e. a discrete-time dynamical system (the continuous-time description will be given later):

\[
\theta(t + \Delta t) = \theta(t) + \int_{t}^{t+\Delta t} D_{B,\theta} \left( \theta(t) \right) dt \tag{2}
\]

\[
r_S(t + \Delta t) = f(\theta(t + \Delta t)) \left[ r_S(t) + \int_{t}^{t+\Delta t} D_{B,r} \left( r_S(t) \cdot f(\theta(t)) \right) dt \right] \tag{3}
\]

\(^1\)From here to the end of this section is the process to obtain the morphed oscillators. If reader is seeking for the final formula, he/she can skip to the end of section 2 and refer to table 1 for a summary and an example.

3
where $D_{B,(f)}(.)$ are the dynamical equations of the base oscillator ($\dot{\theta} = D_{B,\theta}(\theta), r_B = D_{B,r}(r_B)$). Since the base oscillator is a phase oscillator ($\theta = \omega = \text{const}$) we have:

$$D_{B,\theta}(\cdot) = \omega = 2\pi \theta$$

(4)

where $\theta$ is the oscillator’s frequency. Equations (2-3) introduce a general way to convert a chosen base oscillator to one with a desired limit cycle. The shape of the limit cycle in $S$ is determined by $f(\theta)$.

For the oscillator obtained by equations (2-3), the amplitude of the radial trajectories is magnified by the distance from the limit cycle. If $r_B(t) < f(\theta(t))$ then the amplitude of radial trajectories are proportionally reduced, and proportionally enlarged if $r_B(t) > f(\theta(t))$. One might want to have an oscillation behavior which is not magnified by the distance from the limit cycle. To obtain such an oscillator, we can compensate for the magnification effect on $r_B(t + \Delta t)$ if the oscillator state is far from the limit cycle. So we augment equation (3) with a compensation term, $\varepsilon$:

$$r_B(t + \Delta t) = f(\theta(t + \Delta t))\left(\frac{r_B(t)}{f(\theta(t))}\right) + \int_{t}^{t+\Delta t} D_{B,r,\theta}(\frac{r_B(t)}{f(\theta(t))}) dt + \varepsilon$$

(5)

where:

$$\varepsilon = -\left(\frac{r_B(t)}{f(\theta(t))} - \Xi_B(\theta(t))\right)\left(f(\theta(t + \Delta t)) - f(\theta(t))\right)$$

The compensation term $\varepsilon$ is the per-timestep shaping effect scaled by the distance from the limit cycle in $B$. $\varepsilon = 0$ when the oscillator’s state is on the limit cycle, so the shape of the limit cycle is not altered by this term. The map between $S$ and $B$ is still $M(\theta, r) = (\theta, rf(\theta))$, and diffeomorphism still holds. We can simplify equation (5) and rewrite it as:

$$r_B(t + \Delta t) = f(\theta(t + \Delta t))\left(\frac{r_B(t)}{f(\theta(t))}\right) + \int_{t}^{t+\Delta t} D_{B,r,\theta}(\frac{r_B(t)}{f(\theta(t))}) dt + r_B(t) + \Xi_B(\theta(t)) (f(\theta(t + \Delta t)) - f(\theta(t)))$$

(6)

We call the phase oscillator obtained by equations (2,3) the (discrete-time) original form, and the ones obtained by equations (2,6) the (discrete-time) compensated form. Example state-time evolution and phase portraits of both original and compensated forms are depicted in figure 2. Phase portraits in figure 2 show that the limit cycle shape of both forms are the same. The convergence behavior of the original form is scaled by the distance from the limit cycle, so the amplitude of oscillation are reduced when the state is inside (below) the limit cycle and enlarged when state is outside (above) the limit cycle. This means that for a state outside the limit cycle, the distance to the limit cycle might increase in some phases, but will eventually become smaller in at most one cycle (refer to the peaks of the dashed blue trajectories in figure 2). On the other hand, the convergence of the compensated form is such that the distance form the limit cycle is uniformly decreasing in all phases.

As figure 2 depicts, for the compensated form, the $r_B$ state can become negative (figure 2-top, at $t \approx 0.6$) even starting from positive initial values. This means that the space formed by $[\theta, r_B]$ is not exactly the polar coordinates, and negative radius values are meaningful without any shifting to the antiphase state. In this paper we term this space as extended polar coordinates.

2.1. Continuous-time dynamical system

The oscillators obtained by equations (2,6) are with discrete-time updates. A continuous-time dynamical system form can be obtained by calculating the limits of the forward differences of the following equations when $\Delta t \to 0$. For $\theta$ we simply have:

$$\dot{\theta}(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\theta(t) + \left(\int_{t}^{t+\Delta t} D_{B,\theta,\theta}(\theta(t)) dt\right) - \theta(t)\right) = D_{B,\theta,\theta}(\theta(t)) = \omega = 2\pi \theta$$

(7)

which is the definition of the phase dynamics of the base oscillator in $B$. For $r_B(t)$ of the compensated form we have:

$$r_B(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(f(\theta(t + \Delta t))\left(\frac{r_B(t)}{f(\theta(t))}\right) + \int_{t}^{t+\Delta t} D_{B,r,\theta}(\frac{r_B(t)}{f(\theta(t))}) dt + r_B(t) + \Xi_B(\theta(t)) (f(\theta(t + \Delta t)) - f(\theta(t))) - r_B(t)\right)$$

(8)
Figure 2: Comparison between the original and the compensated forms. The base oscillator used is an amplitude controlled oscillator with \( \mu = 1 \) and \( \gamma = 1 \) (please refer to table 2 for the base oscillator’s equations). The desired limit cycle is \( \Xi_0(\theta) = 3 + \tanh(5 \sin(\theta)) + \cos(4\theta + 1) \). The contours of the original form are similar to the limit cycle and scaled by the distance from the limit cycle, while the contours of the compensated form become circular when they are away from the limit cycle. The phase portraits are only for \( r_S > 0 \).

and by simplifying and using Euler approximation for integration we have:

\[
r_S(t) = \lim_{\Delta t \to 0} \left\{ \Xi_0(\theta(t)) \frac{f(\theta(t + \Delta t)) - f(\theta(t))}{\Delta t} + \frac{f(\theta(t + \Delta t)D_{\mu,\nu} \left( \frac{r_S(t)}{f(\theta(t))} \right) \Delta t}{\Delta t} \right\}
\]

and finally:

\[
r_S(t) = \Xi_0(\theta(t)) \frac{f(\theta(t))}{f(\theta(t))} + f(\theta(t))D_{\mu,\nu} \left( \frac{r_S(t)}{f(\theta(t))} \right)
\]

with \( \dot{f}(\theta(t)) = \frac{f(\theta(t))}{f(\theta(t))} = \omega\dot{f}(\theta(t)) \). Equation (7) along with (11) gives a general way to obtain a desired continuous-time phase oscillator. The shape of the limit cycle can be arbitrarily defined by \( \Xi_0(\theta) = \Xi_0(\theta)f(\theta) \). The convergence behavior is defined by the dynamics of the base oscillator.

Same process as in equation (8) can be applied to equation (3) to have the continuous-time dynamical system for the original form. This gives:

\[
r_S(t) = \frac{r_S(t)}{f(\theta(t))} \dot{f}(\theta(t)) + f(\theta(t))D_{\mu,\nu} \left( \frac{r_S(t)}{f(\theta(t))} \right)
\]

The only difference between the obtained equation and equation (11) is the magnification of the canonical term by \( \frac{r_S(t)}{f(\theta(t))} \) instead of \( \Xi_0(\theta(t)) \). This is as we observed with the discrete-time forms of original and compensated forms in figure 2. For the original form, the amplitude of oscillations are magnified by \( \frac{r_S(t)}{f(\theta(t))} \) if the state \( r_S(t) \) is outside the limit cycle. The amplitude of oscillations of the compensated form is not affected by the value of \( r_S(t) \).

The oscillator forms obtained are first order dynamical systems. Positions \( r_S(t) \) and velocities \( \dot{r}_S(t) \) are continuous if not directly perturbed. If \( f \) is additionally \( C^2 \) differentiable, then accelerations \( \ddot{r}_S(t) \) are continuous as well. In general, if \( f \) is \( C^\alpha \) differentiable, consequently \( r_S^{(\alpha)}(t) \) values are continuous, unless directly perturbed.
Compensated form

\[ \dot{\theta} = \omega \]

where a constant obtained by equations (7,12) will follow a desired velocity profile, but yet we need an equation for \( \dot{r}_S \). The desired velocity limit cycle \( \dot{r}_S(\theta) = \dot{r}_S \) is the limit cycle of the base oscillator, and \( f(\theta) \) is the phase-based scaling function. \( \omega \) is the oscillation frequency multiplied by \( 2\pi \). \( \gamma \) controls the rate of convergence. \( \theta \) is the phase of the oscillator, and \( r_S \) is the radial state and also the desired output of the system which converges to the desired limit cycle \( \Xi(\theta) \).

Note: From now on and for the rest of this paper:

- We will use the continuous-time compensated form (equations (7,10)) to extend for higher dynamical system orders and dimension. It is possible to rewrite the same procedures for the original form in a same way.
- We remove the time indexing “\( .(t) \)” for brevity, and will only mention time if there is an explicit time dependency. So terms like \( r_S(\theta) \), \( \dot{\theta}(\theta) \) and \( f(\theta(t)) \) will respectively become \( r_S \), \( \dot{\theta} \) and \( f(\theta) \).
- All the phase portraits are illustrated only for \( r_S > 0 \), unless mentioned otherwise.

2.2. Second order dynamical system

Equations (7, 10) define a first order dynamical system. Equation (7) drives the phase dynamics while equation (10) controls the radial dynamics. If one desires to directly control acceleration, or create the possibility to add feedback mechanisms directly on the acceleration dynamics, then a second order dynamical system is needed. One can rewrite equation (10) for a new velocity state \( v_S \), and take the velocity profile, instead of position profile, as the limit cycle. The desired velocity limit cycle is \( \Xi(\theta) \), so:

\[
\dot{v}_S = \Xi(\theta) \dot{v}_S + f(\theta) D_{B,\gamma} \left( \frac{v_S}{f(\theta) + \delta} \right) + \Xi(\theta) f(\theta) - r_S
\]

where a constant \( \delta > -\min(f(\theta)) \) is added to \( f(\theta) \) to keep the scaling function strictly positive. The oscillator obtained by equations (7,12) will follow a desired velocity profile, but yet we need an equation for \( r_S \). Additionally, unwanted position offsets (errors on \( r_S = \int v_S \, dt \)) are not forgotten in the system defined by equation (12). We add a phase-based attractor force field to \( v_S \) to damp the unwanted position offsets, and attract to the desired limit cycle in \( \Xi \), which is \( \Xi(\theta) = \Xi(\theta) f(\theta) \). By doing so we obtain:

\[
\dot{v}_S = \Xi(\theta) \dot{v}_S + (f(\theta) + \delta) D_{B,\gamma} \left( \frac{v_S}{f(\theta) + \delta} \right) + \beta \left( \Xi(\theta) f(\theta) - r_S \right)
\]

\[
\dot{r}_S = v_S - \delta \Xi(\theta)
\]

Table 1 summarizes the needed formulae to create continuous-time morphed oscillators, and also gives a simple example. Apart from the process to obtain the morphed oscillators which might seem to be complicated, the final formulae has a compact form and is easy to apply. One only needs to define a base oscillator, and choose the desired limit cycle to obtain the desired compact morphed oscillator equations.

<table>
<thead>
<tr>
<th>Morphed oscillator equations</th>
<th>Original form</th>
<th>Compensated form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{\theta} = \omega )</td>
<td>( r_S = \frac{\gamma}{\Xi}(\theta) \dot{f}(\theta) + f(\theta) D_{B,\gamma} \left( \frac{r_S}{\Xi} \right) )</td>
<td>( r_S = \Xi(\theta) \dot{f}(\theta) + f(\theta) D_{B,\gamma} \left( \frac{r_S}{\Xi} \right) )</td>
</tr>
</tbody>
</table>

**Example:** desired limit cycle \( \Xi(\theta) = e^{i\mu(\theta)} \) and base oscillator \( D_{B,\gamma}(r) = \mu(\theta - r) \) (so \( f(\theta) = \frac{\Xi(\theta)}{\Xi(\theta)} = \frac{1}{\mu} e^{i\mu(\theta)} \)).
where $\beta$ determines the strength of the attractor force field defined by $\Xi_B(\theta) f(\theta) - r_S$ term. Also since a $\delta$ offset is added to $f_c$, the velocity limit cycle is having an offset of $\delta \Xi_B(\theta)$, which is subtracted from $v_S$ in equation (14). Equations (7,13,14) together give a general second order phase oscillator which exhibits a desired limit cycle behavior. It should be noted that as long as the base oscillator’s limit cycle is circular $\dot{\Xi}_B(\theta) = 0$, and equations (13,14) can be written as a simpler form where $f_c(\theta) = \dot{f}(\theta)$ and $\dot{f}_c(\theta) = \ddot{f}(\theta)$.

Figure 3 illustrates an example state time evolution and phase portrait of a compensated second-order system when the base oscillator is:

$$\dot{\theta} = 2\pi \dot{\theta}$$  
$$r_S = 2\pi \dot{\theta} \cos(\theta) + 2 + \sin(\theta) - r_B$$  \hspace{1cm} (15)

where this base oscillator has a limit cycle of $\Xi_B(\theta) = 2 + \sin(\theta)$. As the figure shows, the system converges to the desired limit cycle for $r_S$. The desired position and velocity profiles can be crossed by the values of $r_S$ and $v_S$, but the system eventually converges to the desired profiles.

2.3. $n$-th order dynamical system

The idea of extending to a second order system can be generalized to create an $n$-th order dynamical system with an arbitrary limit cycle shape. The deepest state is controlled by the base oscillator, and all states are controlled by
For all the examples given \( \dot{\theta} = 2\pi\theta, r_S = \gamma \tan(\mu - r_B) \). The desired limit cycle is \( \Xi_0(\theta) = 1 + e^{i\theta} \). For this example \( \theta = 1, \mu = 1, \gamma = 10, \beta_1 = \beta_2 = 50, \delta_1 = 50 \) and \( \delta_2 = 150 \).

The previous section gave a general methodology to design morphed phase oscillators with arbitrary limit cycle shapes. The obtained forms represent a general family of morphed oscillators that are parametrized by the radial velocity, and so on. Figure 4 depicts an example state-time evolution for a 3rd order system.

Figure 4: Example state-time evolution of a 3rd order system. The base oscillator is defined with \( \dot{\theta} = f_{n-1}(\theta) + \delta_{n-1} \) \( D_{B_n} \left( \frac{r_S^{(n-1)}}{r_{S_{(n-1)}}} \right) + \beta_{n-1} \Xi_0(\theta) f_{n-2}(\theta) + \delta_{n-2} - r_S^{(n-2)} \) \( r_S^{(n)} = \Xi_0(\theta) f_{n-3}(\theta) + \delta_{n-3} - r_S^{(n-3)} \)

\[ r_S^{(1)} = r_S^{(2)} - \delta_2 + \beta_1 \Xi_0(\theta) f_0(\theta) - r_S^{(1)} \]

\[ r_S = r_S^{(1)} - \delta_1 \]  

(16)

where \( f_{(n)}(\theta) = \frac{d^n \Xi_0(\theta)}{d\theta^n}, \delta_m > \min (f_{(n)}(\theta)), \) and \( \beta_m \) is the strength of the force field on the \( m \)-th order which damps the unwanted offsets of \( r_S^{(m-2)} \). \( r_S^{(1)}, r_S^{(2)}, \ldots, r_S^{(n-1)} \) are system states corresponding respectively to position, velocity, and so on. Figure 4 depicts an example state-time evolution for a 3rd order system.

3. Realization

The previous section gave a general methodology to design morphed phase oscillators with arbitrary limit cycle shapes. The obtained forms represent a general family of morphed oscillators that are parametrized by the radial equation of the base oscillator \( D_{B_n} \). Realizations are obtained by using desired base oscillators, defining \( D_{B_n} \).

Tables 2-3 show first and second order realizations of this methodology with different base oscillators. First a base oscillator of choice is chosen, and then equation (10) or equations (13,14) are used to obtain the morphed oscillator. For all the examples given \( \dot{\theta} = 2\pi\theta, \gamma \) controls the rate of convergence, and \( \mu \) corresponds to the radius of the limit cycle in \( \Xi_0 \).
The first example (I) in tables 2-3 uses an amplitude controlled oscillator as base. The $r_S$ equation is linear to $r_S$, and the base oscillator has a limit cycle at $\Xi(\theta) = \mu$. The second example (II) is when the base oscillator is a Hopf oscillator. For this oscillator $\Xi(\theta) = \sqrt{\mu}$. The diffusive term is a cubic polynomial, and it becomes 0 when $r_S^4 = \pm \sqrt{\mu}$. Example (III) uses a base oscillator with a logarithmic saturated convergence rate. For this base, $\Xi(\theta) = \sqrt{\mu}$, which is similar to (II). The fourth example (IV) is the case where a morphed oscillator is used as base, which itself is a morphed amplitude controlled oscillator with a limit cycle shape of $\Xi(\theta) = 2 + \sin(\theta)$. For all these oscillators the shape of the oscillator in $\mathbb{S}$ is $\Xi_S(\theta) = \Xi_S(\theta)f(\theta)$.

Figure 5 illustrates phase portraits of these oscillators as well as their state-time evolution (both for first order realizations). The desired limit cycle $\Xi_S(\theta)$ is chosen to be the same for all the examples to enable comparison. It is clear that the difference is in their convergence behavior and basin of attraction, and the shape of the limit cycle is the same, as chosen. One should keep in mind that although $\Xi_S(\theta)$ is the same for all the examples, the corresponding $f(\theta)$ functions are not necessarily the same, because $\Xi_S(\theta)$ is different for different base oscillators (and we know $f(\theta) = \frac{\Xi_S(\theta)}{\Xi_S(\theta)}$).

The basin of attraction in examples (I) and (IV) is the whole state space, while this is not true for examples (II) and (III). Example (II) uses a Hopf oscillator as base, which has two limit cycles at $r_S(\theta) = \pm \sqrt{\mu}$, and states who enter the basin of attraction of the unwanted limit cycle at $r_S(\theta) = -\sqrt{\mu}$ can get trapped. Finally, for example (III), the solution diverges initiating with $r_S|_{t=0} = -2$. We will discuss the stability conditions in section 5 and see when the oscillator in example (III) gets unstable.

3.1. Equivalence

Here we will show how certain realizations of the morphed oscillator represent familiar trajectory control/generation methods. The first example in Table 2 uses an amplitude controlled oscillator as base. Assuming $\mu = 1$, the first ex-
### Base oscillator | Compensated first order realization
---|---
(I) Amplitude controlled oscillator \( r_\beta = \gamma (\mu - r_\beta) \) | \( r_\beta = \mu f(\theta) + \gamma (\mu f(\theta) - r_\beta) \)

(II) Hopf oscillator \( r_\beta = \gamma (\mu - r_\beta^2) r_\beta \) | \( r_\beta = \sqrt{\mu f(\theta)} + \gamma \left( \mu - \left( \frac{r_\beta}{\mu} \right)^2 \right) r_\beta \)

(III) Logarithmic saturated oscillator \( r_\beta = \gamma \tanh(\mu - r_\beta^2) \log(1 + r_\beta^2) \) | \( r_\beta = \sqrt{\mu f(\theta)} + \gamma f(\theta) \tanh \left( \mu - \left( \frac{r_\beta}{\mu} \right)^2 \right) \log \left( 1 + \left( \frac{r_\beta}{\mu} \right)^2 \right) \)

(IV) Morphed oscillator \( r_\beta = \omega \cos(\theta) + 2 + \sin(\theta) - r_\beta \) | \( \dot{r}_\beta = (2 + \sin(\theta)) f(\theta) + f(\theta) \left( \omega \cos(\theta) + 2 + \sin(\theta) - \frac{r_\beta}{\mu} \right) \)

#### Table 2: Examples first order realizations with different base oscillators.

### Base oscillator | Compensated second order realization
---|---
(I) Amplitude controlled oscillator \( r_\beta = \gamma (\mu - r_\beta) \) | \( \ddot{r}_\beta = \mu f(\theta) + \gamma (\mu f(\theta) - r_\beta) + \beta (\mu f(\theta) - r_\beta) \)

(II) Hopf oscillator \( r_\beta = \gamma (\mu - r_\beta^2) r_\beta \) | \( \ddot{r}_\beta = \sqrt{\mu f(\theta)} + \gamma (\mu - \left( \frac{r_\beta}{\mu} \right)^2) r_\beta \)

(III) Logarithmic saturated oscillator \( r_\beta = \gamma \tanh(\mu - r_\beta^2) \log(1 + r_\beta^2) \) | \( \ddot{r}_\beta = \sqrt{\mu f(\theta)} + \gamma f(\theta) \tanh \left( \mu - \left( \frac{r_\beta}{\mu} \right)^2 \right) \log \left( 1 + \left( \frac{r_\beta}{\mu} \right)^2 \right) \)

(IV) Morphed oscillator \( r_\beta = \omega \cos(\theta) + 2 + \sin(\theta) - r_\beta \) | \( \ddot{r}_\beta = (2 + \sin(\theta)) f(\theta) + f(\theta) \left( \omega \cos(\theta) + 2 + \sin(\theta) - \frac{r_\beta}{\mu} \right) \)

#### Table 3: Examples second order realizations with different base oscillators.
ample implements a position PD-controller with a desired periodic reference defined as $\Xi_\varphi(\theta) = \Xi_B(\theta)f(\theta) = f(\theta)$. The normalized proportional gain for this controller is $\gamma$. So with the right representation, position PD-control over a periodic reference is a subset of the general family generated by the compensated form.

The first example in Table 3 implements a second order dynamical system which represents a form of the rhythmic Dynamical Movement Primitives [26]. Dynamical Movement Primitives (DMP) are robust movement generators that are commonly used for motor control. The rhythmic DMP is formulated as:

$$
\tau \ddot{z} = \alpha_z(\beta_z(g - y) - z) + F
$$

$$
\tau \dot{y} = z
$$

$$
\tau \dot{\theta} = 1
$$

$$
F = \tau^2 \dot{y} y_{des}(t) + \tau \alpha \beta y_{des}(t) + \alpha \beta y_{des}(t) - \alpha \beta g
$$

$$
= \tau^2 \left( \frac{1}{\tau} \right) y_{des}(\theta) + \tau \left( \frac{1}{\tau} \right) \alpha \beta y_{des}(\theta) + \alpha \beta y_{des}(\theta) - \alpha \beta g
$$

where the set of equations (17,18) is called the transformation system, equation (19) is called the canonical system, $\tau$ is the cycle period divided by $2\pi$, and $F$ is the nonlinear forcing term to shape the limit cycle such that the output $y_{des}$ is obtained. Rewriting the transformation system with expanded $F$ and simplifying gives:

$$
\dot{z} = \tau \dot{y} y_{des}(\theta) + \frac{\alpha_z}{\tau}(\tau \dot{y} y_{des}(\theta) - z) + \frac{\alpha \beta y_{des}(\theta)}{\tau}(y_{des}(\theta) - y)
$$

$$
\dot{y} = \frac{1}{\tau} z
$$

and by assuming $r_B = y$, $v_B = \frac{1}{2} \dot{z}$ and $f(\theta) = y_{demo}(\theta)$, above is identical to a second-order realization with a unit radius amplitude controlled oscillator as base ($\mu = 1$), and $y = \frac{1}{2}$ and $\beta = \frac{\alpha \beta}{\tau}$. So with the right representation, rhythmic DMPs (with the latest representation in [26]) are a subset of the general family generated by the compensated form.

4. Extension to higher dimensions

The methodology presented in section 2 gives a systematic way to create nonlinear oscillators with one dimensional outputs $r_B$. In this section we explain how multidimensional oscillators can be created out of these one dimensional oscillators. Extending to high dimensions expands the scope which the introduced oscillators can be applied, including coupled synchronized high dimensional movements needed in applications like robotics [27, 28], or in abstract modeling of neuroscientific models of Central Pattern Generators [3, 4].

Several approaches including [18, 29, 13] use coupled oscillators to create multidimensional nonlinear oscillators. One can use diffusive phase coupling [30] to create the multidimensional system, so equation (7) changes to:

$$
\theta_i = \frac{2\pi}{\tau} \theta_i + \sum_{k=1}^{N} c_{ik} \sin(\theta_k - \theta_i - \phi_{ik})
$$

where $\theta_i$ is the phase state of the $i$–th oscillator, $c_{ik}$ is the coupling strength between $i$–th and $k$–th oscillators, and $\phi_{ik}$ is the desired phase difference between them. As long as the $\theta_i$ frequencies are equal, coupling strengths are non-negative, and coupling phase-differences are consistent, the coupled system asymptotically converges to the desired phase differences. This will be further discussed in section 5. The diffusive schema in equation (22) couples the phase dynamics, and this coupling is independent of the output states $r_{E,j}$. Implementing coupling schema which are affected by the $r_{E,j}$, states is possible, and is application specific. An example can be found in [29].

Figure 6 illustrates an all-to-all coupled four dimensional system where each oscillator is a second-order morphed oscillator. A different limit cycle shape is chosen for each dimension, and these are depicted in the top four plots. The bottom-left plot shows the phase-time evolution, and phase differences can be seen with the overlaid markers, where the desired phase differences, with respect to $\theta_1$, are $[0, \pi/3, \pi/2, \pi]$. All phases are initialized with 0 value, and they are also perturbed for $i \in [2, 2.1]$. The system gradually gets synchronized after the perturbation, as illustrated in the bottom-right plot in figure 6.
5. Stability

This section is dedicated to stability analysis of the introduced morphed oscillators. We first discuss the stability of a one dimensional first order system using the Poincaré-Bendixson theorem [31]. Analysis of $n$--th order systems, $n > 1$, is complex as the Poincaré-Bendixson theorem is not valid anymore, and we briefly discuss these systems using contraction theory [32]. Finally we include the stability of the multidimensional system.

Before going into the stability analysis, we would like to describe the space formed by $[\theta, r_3]$. One can simply assume that $\theta \in [0, 2\pi)$, $r_3 \in \mathbb{R}^+$, and by doing so $[\theta, r_3]$ form the standard polar coordinates. However we additionally want to be able to analyze the system when $r_3 < 0$ (without jumping to the antiphase state $[\theta + \pi, |r_3|]$). One can assume that $\theta \in [0, 2\pi)$, $r_3 \in \mathbb{R}^+$ forms a manifold, and $\theta \in [0, 2\pi)$, $r_3 \in \mathbb{R}^-$ forms a second manifold, and these two manifolds are connected when $r_3 = 0$. The resulting manifold can be chosen to be a 2-manifold, as illustrated in figure 7, to describe the space formed by $[\theta, r_3] \in [0, 2\pi) \times \mathbb{R}$. We call this representation the extended polar coordinates where negative radius values are meaningful. The chosen 2-manifold is only an arbitrary representation, and any other representation which defines an orientable 2-manifold is valid.

5.1. One dimensional first order system

Analyzing the limit cycle properties of a dynamical system is not an easy task. The analysis becomes possible when one is looking for the existence of limit cycles in a bounded region of a phase plane, where the Poincaré-Bendixson’s theorem can be utilized. The same theorem can also be used the analyze the asymptotic stability properties of a limit cycle system. The original form of this theorem is:

**Theorem** (Poincaré-Bendixson [31]). Every nonempty, compact $\omega$-limit set of a $C^1$ planar flow that does not contain an equilibrium point is a (nondegenerate) periodic orbit.

A simpler interpretation of this theorem is: given a differential equation in the plane, assume $\zeta(t)$ is a solution curve which stays in a bounded region. Then, if there is no equilibrium point in this region, $\zeta(t)$ converges for $t \to +\infty$ to a
periodic trajectory. Now if there is only one periodic orbit in this region, the asymptotic stability of the corresponding limit cycle in this bounded region is assured.

Poincaré-Bendixson’s theorem also holds for every orientable 2-manifold for which the Jordan curve theorem [33] holds. The extended polar coordinates defined by \( [\theta, r_S] \) is an orthogonal coordinate system, and the manifold created by \( [\theta, r_S] \in \mathbb{R}^2 \) is an orientable 2-manifold satisfying Jordan curve theorem.

To employ Poincaré-Bendixson’s theorem on our problem, we need to define a bounded region around the desired limit cycle \( \Xi_B(\theta) \). We define the Lower and Upper bounds around \( \Xi_B(\theta) \) as:

\[
\begin{align*}
g_L : \theta & \mapsto g_L(\theta) \colon g_L(\theta) < \Xi_B(\theta) \\
g_U : \theta & \mapsto g_U(\theta) \colon g_U(\theta) > \Xi_B(\theta)
\end{align*}
\]

For the upper bound, vectors pointing inwards the bounded region are defined as \( p = \{\dot{g}_U(\theta), -\dot{\theta}\} \), and for the lower bound they are \( p = \{-\dot{g}_L(\theta), \dot{\theta}\} \) (assuming clockwise phase evolution, i.e. \( \dot{\theta} > 0 \)). The system dynamics on these bounds are also defined as:

original form, equation (11) :
\[
d = \{\dot{\theta}, \frac{g_L(\theta)}{f(\theta)} + f(\theta)D_B, r(\Xi_B(\theta) + \kappa)\}, \theta \in \{L, U\}
\]

compensated form, equation (10) :
\[
d = \{\dot{\theta}, \Xi_B(\theta)f(\theta) + f(\theta)D_B, r(\frac{g_L(\theta)}{f(\theta)})\}, \theta \in \{L, U\}
\]

To have the condition that no flow leaves the bounds we need (with \( \langle ., . \rangle \) operator being the inner product):

\[
(p, d) > 0
\]

To utilize the Poincaré-Bendixson’s theorem, it is needed to define bounds such that no flows leaves the enclosed area. Figure 8 gives an idea about how to define the bounds. For the original form we define:

\[
g_{LU}(\theta) = \Xi_B(\theta) + \kappa f(\theta) = \Xi_B(\theta) + \kappa f(\theta); \kappa \leq 0
\]

Rewriting equation (25) for the upper bound gives:

\[
\dot{\theta}(\Xi_B(\theta)f(\theta) + (\Xi_B(\theta) + \kappa)f(\theta)) - \dot{\theta}(\Xi_B(\theta) + \kappa)f(\theta)f(\theta) - \dot{\theta}f(\theta)D_B, r(\frac{(\Xi_B(\theta) + \kappa)f(\theta)}{f(\theta)}) > 0
\]

and dividing by the positive term \( \dot{\theta}f(\theta) \) gives (by definition \( f(\theta) > 0 \) and we have already assumed \( \dot{\theta} > 0 \)):

\[
D_B, r(\Xi_B(\theta) + \kappa) < \Xi_B(\theta); \kappa > 0
\]
and for the lower bound we similarly obtain:

\[ D_{B_J}(\Xi_b(\theta) + \kappa) > \Xi_b(\theta); \kappa < 0 \]  

(29)

Now if the conditions in equations (28-29) are met for all possible margins in a bounded region, then the desired limit cycle is asymptotically stable. So the basin of attraction for the original form is defined by the bounds:

\[ \kappa_L \in \mathbb{R} \mid \forall \theta \in \mathbb{R} & \forall \kappa, \kappa_L < \kappa < 0 : D_{B_J}(\Xi_b(\theta) + \kappa) > \Xi_b(\theta) \]

\[ \kappa_U \in \mathbb{R} \mid \forall \theta \in \mathbb{R} & \forall \kappa, 0 < \kappa < \kappa_U : D_{B_J}(\Xi_b(\theta) + \kappa) < \Xi_b(\theta) \]  

(30)

The lower and upper bounds for the compensated form in equation (10) are not same as the ones of the original form. This is depicted in figure 8. What happens with the compensated form is that the dynamics behave like the shape of limit cycle for states near it, but behaves as non-shaped exponentials for states far from limit cycle. This means that the bound should be similar to the limit cycle’s shape for states very near the limit cycle, and should be uniformly circular when the states are infinitely far from the limit cycle. So we define:

\[ g_{LU}(\theta) = \Xi_b(\theta) + \kappa = \Xi_b(\theta)f(\theta) + \kappa; \kappa \leq 0 \]  

(31)

and by using the same procedure as used for the original form we get:

\[ \text{lower bound} \implies \mp \dot{b} \left( \Xi_b(\theta)f(\theta) + \Xi_b(\theta)f'(\theta) \right) \pm \theta \Xi_b(\theta)f'(\theta) \pm \dot{b} f(\theta) D_{B_J}(\Xi_b(\theta)f(\theta) + \kappa) > 0 \]

\[ \implies D_{B_J}(\Xi_b(\theta) + \frac{\kappa}{f(\theta)}) \geq \Xi_b(\theta); \kappa \leq 0 \]  

(32)

The term \( \frac{\kappa}{f(\theta)} \) is strictly positive when \( \kappa > 0 \) and strictly negative if \( \kappa < 0 \). So again the basin of attraction is defined by the bounds:

\[ \kappa_L \in \mathbb{R} \mid \forall \theta \in \mathbb{R} & \forall \kappa, \kappa_L < \kappa < 0 : D_{B_J}(\Xi_b(\theta) + \frac{\kappa}{f(\theta)}) > \Xi_b(\theta) \]

\[ \kappa_U \in \mathbb{R} \mid \forall \theta \in \mathbb{R} & \forall \kappa, 0 < \kappa < \kappa_U : D_{B_J}(\Xi_b(\theta) + \frac{\kappa}{f(\theta)}) < \Xi_b(\theta) \]  

(33)
Table 4: The basin of attraction of first order realizations using different bases.

<table>
<thead>
<tr>
<th>base oscillator</th>
<th>basin of attraction bounds parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>original</td>
</tr>
<tr>
<td>$\theta = \omega$</td>
<td>LB: $\kappa \rightarrow -\infty$</td>
</tr>
<tr>
<td>$r_\theta = \gamma(\mu - r_\theta)$</td>
<td>LB: $\kappa = -\sqrt{\mu}$</td>
</tr>
<tr>
<td>$\theta = \omega$</td>
<td>LB: $\kappa = -\sqrt{\mu}$</td>
</tr>
<tr>
<td>$r_\theta = \gamma(\mu - r_\theta^2)r_\theta$</td>
<td>LB: $\kappa = -\sqrt{\mu}$</td>
</tr>
<tr>
<td>$\theta = \omega$</td>
<td>LB: $\kappa = -2\sqrt{\mu}$</td>
</tr>
<tr>
<td>$r_\theta = \omega \cos(\theta) + 2 + \sin(\theta) - r_\theta$</td>
<td>LB: $\kappa = -\infty$</td>
</tr>
<tr>
<td>$\theta = \omega$</td>
<td>LB: $\kappa = -\mu$</td>
</tr>
<tr>
<td>$r_\theta = -\gamma r_\theta(\mu - r_\theta)(2\mu - r_\theta)$</td>
<td>LB: $\kappa = -\mu \min(1/f(\theta))$</td>
</tr>
</tbody>
</table>

Table 4 gives examples of the basin of attraction for different realizations which are obtained using equations (30,33). The first four examples are the same morphed oscillators as in table 2. The fifth example is with a base oscillator that has one stable limit cycle at $r_{\theta} = \mu$ and two unstable ones at $r_{\theta} = 0$ and $r_{\theta} = 2\mu$. For the first four examples, the upper bound of the basin of attraction tends to infinity. The basin of attraction for the fifth example is bounded by the two other unstable limit cycles.

We should mention that the basin of attraction described in equation (33) does not necessarily give the whole basin of attraction (for the compensated form). Assuming a Hopf oscillator, for $-\sqrt{\mu} < r_{\theta} < 0$ the base oscillator is not converging toward the limit cycle at $r_{\theta} = \sqrt{\mu}$, and is instead converging towards the one at $r_{\theta} = -\sqrt{\mu}$. But if the magnitude of the dynamics in $-\sqrt{\mu} < r_{\theta} < 0$ is small compared to $f(\theta)$, then the dynamical system can escape this region, and converge to the limit cycle at $r_{\theta} = \sqrt{\mu}$. Figure 9 shows the case where the Hopf oscillator has a small $\mu = 1$ and the dynamics can escape the trap region, where for $\mu = 10$ the dynamics gets trapped and cannot converge to the desired limit cycle at $r_{\theta} = \sqrt{\mu}$. There is a numerically estimated bifurcation around $\gamma = 4.85$. So, for $\gamma < 4.85$, even tough the initial value of $r_{\theta} = -2.5$ is not within the basin of attraction obtained from equation (33), the orbit converges to the desired limit cycle.

5.2. One dimensional second order system

Analyzing the asymptotic stability of a general nonlinear second order system is not trivial, and the Poincaré-Bendixson theorem cannot be used anymore as it is only valid for phase planes, and not volumes. We utilize Contraction Theory [32] to show the stability conditions of our 2nd+ order systems. Contraction Theory provides a general
method for the analysis of nonlinear systems by a transfer to composite systems, which makes it suitable for the analysis of complex systems. Contraction Theory characterizes the system stability by the behavior of the differences between solutions with different initial conditions. If these differences vanish exponentially over time, all solutions converge towards a single trajectory, independent from the initial states. In this case, the system is called globally asymptotically stable. For a general dynamical system of the form:

\[ \dot{x} = f(x, t) \]  

(34)

assume that \( x(t) \) is one solution of the system, and \( \hat{x}(t) = x(t) + \delta x(t) \) a neighboring one with a different initial condition (\( \delta x(t) \) is also called virtual displacement). It can be shown that any nonzero virtual displacement decays exponentially to zero over time if the symmetric part of the Jacobian of equation (34) is uniformly negative definite. In this case, it can be shown that the norm of the virtual displacement decays at least exponentially to zero, for \( t \to \infty \) [32]. Namely, \( \| \delta x \| \leq \exp \left( \int_{t_0}^{t} \lambda_{\text{max}}(x, t) dt \right) \| \delta x_0 \| \), where \( \lambda_{\text{max}}(x, t) \) is the largest eigenvalue of \( \frac{1}{2} (\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^T) \), so that the sufficient contraction condition is:

\[ \lambda_{\text{max}}(x, t) dt \leq -b < 0 \]  

(35)

where \( J \) is the unsymmetrized Jacobian of equations (13,14). This has a symmetrized Jacobian:

\[ J_{\text{sym}} = \begin{bmatrix} D_{r,f}(1) & (1-\beta)/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]  

(36)

which has a zero eigenvalue and the other one is \( D_{r,f}(\sqrt{\beta}) \). So the second order system is contracting when \( D_{r,f}(\beta) < 0, \beta > 0 \). With the above conditions, the compensated second order form is partially contracting towards the desired limit cycle and the phase subsystem is indifferent.

5.3. One dimensional \( n \)-th order system

The stability analysis of the \( n \)-th order system \( (n > 2) \) is similar to the one for a 2nd order system. Again writing the infinitesimal virtual displacements give:

\[ \frac{d}{dt} \begin{bmatrix} \delta r_{(n-1)}^{(n-2)} \\ \delta r_{(n-2)}^{(n-3)} \\ \vdots \\ \delta r_{(1)}^{(n-2)} \\ \delta r_{\delta}^{(1)} \end{bmatrix} = \begin{bmatrix} D_{r,f}^{(n-1)}(1) & -\beta_{n-1} & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & 1 & 0 & -\beta_1 & \delta r_{\delta}^{(1)} \\ 0 & \ldots & 0 & 0 & 1 & 0 & \delta r_{\delta}^{(1)} \end{bmatrix} \]  

(37)

and again using the new coordinates: \( \delta r_{\delta}^{(n-1)}, \delta r_{\delta}^{(n-2)} = \sqrt{\beta_{n-1}} \delta r_{\delta}^{(n-2)}, \delta r_{\delta}^{(n-3)} = \sqrt{\beta_{n-2}} \delta r_{\delta}^{(n-3)}, \ldots, \delta r_{\delta}^{(1)} = \sqrt{\beta_1} \delta r_{\delta}^{(1)} \), the \( n \)-th order system is partially contracting when \( D_{r,f}^{(1)}(1) < 0 \) and \( \forall i : \beta_i > 0 \).
5.4. Multidimensional system

The stability of the multidimensional system created in equation (22) should be analyzed from two aspects: (I) asymptotic stability of the phase coupling, (II) asymptotic stability of each dimension under coupling. Since phases are independent of the radial values, the multidimensional system is a hierarchy, and the asymptotic stability of phases can be analyzed independently [32]. To address (I), it can be shown that as long as for every loop in the coupling graph passing through oscillators \(i_1, i_2, i_3, \ldots, i_m, i_l\):

\[
\phi_{i_1i_2} + \phi_{i_2i_3} + \ldots + \phi_{i_mi_l} = 2k\pi, \ k \in \mathbb{Z}
\]

(39)

and with the additional conditions \(\forall i, j : c_{ij} \geq 0\) and \(\forall i, j : \phi_i = \phi_j\), the phase differences \(\theta_i - \theta_j\) will asymptotically converge to the desired phase differences \(\phi_{ij}\). One can introduce the potential function (for the coupling dynamics in equation (22)):

\[
U(\theta) = -\sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} \cos(\theta_j - \theta_i - \phi_{ij})
\]

(40)

which gives: \(\dot{\theta}_i = \frac{\partial U}{\partial \theta_i}\) and the potential \(U(\theta)\) has minima at \(\forall i, j : \theta_i = \theta_j - \phi_{ij} + 2k\pi, \ k \in \mathbb{Z}\). Since \(\frac{dU}{d\theta_i} = \sum_{j=1}^{N} \frac{\partial U}{\partial \theta_j} \frac{\partial \theta_j}{\partial \theta_i} = -\sum_{j=1}^{N} \frac{\partial U}{\partial \theta_j} \frac{\partial \theta_j}{\partial \theta_i}\), then \(U(\theta)\) plays a role of Lyapunov’s function, proving the asymptotic stability. Now since the phase differences are consistent (equation (39)), the system remains synchronized.

The multidimensional system is a hierarchically coupled system, where radial dynamics depends on phase dynamics but phase dynamic does not depend on radial dynamics, and the phase dynamics is contracting by itself. As long as the phases are synchronized, \(\forall i : \dot{\theta}_i = \omega = 2\pi \theta\), then all oscillators will converge to their limit cycles. An asynchrony, e.g. \(\theta_j - \theta_i \neq 2k\pi + \phi_{ij}\), can introduce a perturbation on the radial dynamics for the \(i\)-th dimension. Let us assume that this perturbation, at phase \(\theta_i\), is quantified as \(u_i(\theta_i)\). As long as this perturbation does not push the \(i\)-th oscillator out of its basin of attraction, it will eventually be forgotten, and the \(i\)-th oscillator will converge to its limit cycle. So in the case where the basin of attraction is the whole state space, the whole multidimensional system is asymptotically stable. For the case where the basin of attraction is limited, assuming that the perturbation happens in a synchronized state, if:

\[
\begin{align*}
\min_{r \in [0, \mathbb{R} \mathbb{Z}^{n-1}(\theta)]} \left( f_{(n-1)}(\theta_i) + \delta(\theta_i - \phi_{ij}) \right) \mathcal{D}_{\Theta} \left( \sum_{k=1}^{N} c_{ik} \right) \\
\max_{r \in [0, \mathbb{R} \mathbb{Z}^{n-1}(\theta)]} \left( f_{(n-1)}(\theta_i) + \delta(\theta_i - \phi_{ij}) \right) \mathcal{D}_{\Theta} \left( \sum_{k=1}^{N} c_{ik} \right)
\end{align*}
\]

(41)

then the perturbation will not increase the radial distance of the system state to the limit cycle \(\mathbb{Z}_{n-1}^{(n-1)}(\theta)\), thus the system will remain in the basin of attraction. Now if we estimate the maximal \(u_i(\theta)\) as:

\[
|u_i(\theta)| = \max_{\theta} \left( f_{(n-1)}(\theta) \right) \sum_{k=1}^{N} c_{ik}
\]

(42)

then, if the sum of the coupling weights is small enough, the system will remain in the basin of attraction. One can simply choose to have all the coupling weights equal and small enough to ensure stability.

6. Arbitrary convergence behavior

The previous sections introduced a methodology to design morphed phase oscillators with arbitrary limit cycle shapes, and analyzed their stability. The convergence behavior of these morphed oscillators is determined by the choice of the base oscillator, and cannot be explicitly defined. However it is useful to have an oscillator which can exhibit a given desired convergence behavior. Examples can be when a certain path should be followed to reach the limit cycle, or if phase-dependent convergence behavior is of interest.
To obtain such oscillator, we modify the simple case where an amplitude controlled oscillator is used as the base oscillator and an original first order realization is applied. We first need to find the analytical solution for the case where an amplitude controlled oscillator is used as the base. For the phase equation we can simply write:

$$\theta(t) = \omega t + C_\theta, \; \theta(0)$$  \hspace{1cm} (43)

To find the analytical solution for the radius equation, we define $g = f(\theta)$, and we rewrite the morphed amplitude controlled oscillator (first example in table 2) as (for simplicity, and without loss of generality, we assume $\mu = 1$):

$$\dot{r}_2 + \left( \frac{\gamma - \frac{h}{R}}{g} \right) r_2 = \gamma g$$  \hspace{1cm} (44)

where $\dot{g} = \omega f'(\theta)$. This is a first order differential equation of the form:

$$\dot{r}_2 + pr_2 = q$$  \hspace{1cm} (45)

with $p = \gamma - \frac{h}{R}$ and $q = \gamma g$. This form has the general solution: $r_2 e^{\int p dt} = \int q e^{\int p dt} dt + C_r$. Solving the integration with respective $p$ and $q$ expressions gives (with $g = f(\theta)$):

$$r_2(t) = f(\theta(t)) + C_r f(\theta(t)) e^{-\gamma t}, \; C_r = \frac{r_2(0) - f(\theta(0))}{h(0)} - 1$$  \hspace{1cm} (46)

The analytical solution shows that the oscillator converges to the desired limit cycle behavior $\Xi_2(\theta) = f(\theta)$ when $t \to \infty$. The convergence behavior is an exponential decay which is shaped by $f$. Now if we use a custom convergence function $h$ instead of $f$ to define the convergence behavior, we can modify equation (46) to represent the new desired solution:

$$r_2(t) = f(\theta(t)) + C_r h(t) e^{-\gamma t}, \; C_r = \frac{r_2(0) - f(\theta(0))}{h(0)}$$  \hspace{1cm} (47)

To obtain the dynamical system yielding this desired solution we perform the steps to obtain equation (46) backwards. If we multiply both sides of equation (47) with $\frac{1}{h}$ we have (again $g = f(\theta)$): $r_2 \frac{d}{dt} = e^{\int \frac{\gamma}{h} dt} + C_r$, which can be rewritten as:

$$r_2 \frac{e^{\int \frac{\gamma}{h} dt}}{h} = \int e^{\int \frac{\gamma}{h} dt} \left( \gamma g + \frac{\dot{h}h - h\dot{g}}{h} \right) dt + C_r$$  \hspace{1cm} (48)

Now if we define $e^\nu = e^{\int \frac{\gamma}{h} dt}$ we obtain the general coefficients $p = \gamma - \frac{h}{R}$ and $q = \gamma g + \frac{\dot{h}h - h\dot{g}}{h}$. With $p$ and $q$ defined, equation (48) is a solution of a first order differential equation of the form defined in equation (45). We obtain:

$$\dot{r}_2 + \left( \frac{\gamma - \frac{h}{h}}{h} \right) r_2 = \gamma g + \frac{\dot{h}h - h\dot{g}}{h}$$  \hspace{1cm} (49)

and we can simplify and write it as an ordinary differential equation for radius (with $g = f(\theta)$):

$$\dot{r}_2 = \omega f'(\theta) + \left( \gamma - \frac{\dot{h}(\theta, t)}{h(\theta, t)} \right) (f(\theta) - r_2)$$  \hspace{1cm} (50)

Equation (50) along with the phase equation (7) gives a first order morphed phase oscillator which the shape of its limit cycle is defined by $f$ and the shape of its convergence by $h$. As the desired solution in equation (47) shows, the steady state solution does not depend on $h$ and is only defined by $f$. The function $h(\theta, t)$ can be a function of phase, time, or both, and depends on the application. One can argue that the term $\gamma - \frac{\dot{h}(\theta, t)}{h(\theta, t)}$ can be generally written as $\gamma(\theta, t)$ which means having a phase - and/or time - dependent convergence rate. This is correct, however a representation like $\gamma(\theta, t)$ will not explain what the convergence behavior will be.

To ensure a stable system, the fixed-point $f(\theta) - r_2$ in equation (50) should be attractive. This means the condition $\gamma - \frac{\dot{h}(\theta, t)}{h(\theta, t)} > 0$ should be satisfied. Moreover, since $h(\theta, t)$ is in the denominator in equation (50), $h(\theta, t)$ should not have a zero-crossing.

Figure 10 shows examples of the arbitrary convergence behavior. The top plot in figure 10 depicts the effect of perturbation on different $h$ functions. The left plot in figure 10 shows the phase portrait of a system with four different convergence behaviors. As it is clear, the limit cycle shape is not affected by the choice of the convergence behavior.
7. Learning

Previous sections have all used scaling functions that are defined as closed-form functions. This section will explain how the scaling function $f$ can be created from data points. We will additionally explain how the convergence behavior $h$ can be fitted from data when equation (50) is of interest. All the given descriptions are for one-dimensional cases, and extension to multi-dimensional cases is done by just repeating the same process for all dimensions and setting correct phase differences $\phi_{ij}$.

7.1. Learning the scaling function

The scaling function along the shape of the limit cycle in $\mathbb{B}$ together define the shape of the limit cycle is $S$. Let us assume that one dimensional input data is given as $\{t_i, y_i\}, i = 1 \ldots N$, where $t_i$ is the sample time, $y_i$ is the desired output, and $N$ is the number of data points. This data vector should represent a periodic activity. We first need to extract the frequency of oscillation, which methods like discrete Fourier transform or cross correlation can be used. After the frequency $\vartheta$ is determined, we create the phase data as:

$$\theta_i = 2\pi \vartheta t_i$$  \hspace{1cm} (51)

We can then use the values of $y_i$ as the desired radius of the limit cycle. We add a constant offset $\delta_0$ to $y_i$ values in case they include negative values. This is due to the fact that $f$ should be a positive function (other than the case where a base oscillator with a linear $r_B$ equation is used), and the same $\delta_0$ should be subtracted when reading the output of the system. Consequently the data describing the scaling function $f$ is defined as (with $\delta_0 > -\min_i(y_i)$):

$$f : \theta_i \longmapsto \frac{y_i + \delta_0}{\Xi_B(\theta_i)}, \text{ } i = 1 \ldots N$$  \hspace{1cm} (52)

The dataset given in equation (52) can be used to create any function approximator describing $f$, as long as it keeps the periodicity with a period of $1/\vartheta$. As recommended by [26], the shaping function can be modeled with normalized
weighted periodic Gaussian-like bases, known as von Mises basis functions:

\[
f(\theta) = \frac{\sum_{k=1}^{K} w_k \psi_k(\theta)}{\sum_{k=1}^{K} \psi_k(\theta)}
\]

\[
\psi_k(\theta) = e^{\gamma (\cos(\theta - c_k) - 1)}
\]

where \( \psi_k \) is a von Mises basis centered at phase \( c_k \) and \( \sigma_k \) determines the span. If \( f \) is modeled so, then there are powerful tools like locally weighted regression [34] to find the \( w_k \) parameters in an \( O(KN) \) procedure. One additional benefit of using von Mises bases is that the resulting \( f \) function is smooth, i.e. it is \( C^\infty \) differentiable. This means that, even using the first order realization, all position, velocity, acceleration, etc states will be continuous, until perturbed.

We like to mention that one nice outcome of using a mixture of periodic basis functions to model \( f \) is that they can represent \( f \) in terms of simple movement/motor primitives, which can be linked to more biological explanation of how movements are coded and generated [35–37]. Rhythmic Dynamical Movement Primitives [26] are one example of such, and the framework here is a superset of rhythmic DMPs, and the notion of movement primitives apply as long as the \( f \) function is accordingly represented.

7.2. Learning the convergence function

In case where a morphed oscillator having a desired limit cycle shape but also with an explicit desired convergence behavior is of interest, one would use equations (7,50) to model it. Let us assume that the given data \( \{t_i, y_i\}, i = 1 \ldots N \) describes the oscillation behavior from an initial condition \( y_0 \) which convergences to a periodic behavior\(^2\). We can simply take the last periods of oscillation (where the oscillator is already converged enough with respect to an error measure), and use this part to model \( f \) as detailed in the last section. Knowing \( f \), and by utilizing equation (47), we have:

\[
y_i + \delta_0 = f(\theta_i) + \frac{y_0 - f(\theta_0)}{h(0)} h(t) e^{-\gamma t}
\]

\(^2\)Our approach here is limited to having one single example. If multiple examples are given, one can first apply the process explained for one example, and then use the median of the resulting parameters.
which can be rewritten as:

\[ h(t, \theta) e^{-\gamma t} = \frac{h(t)}{h(0)} e^{-\gamma t} = \frac{y_t + \delta_0 - f(\theta)}{y_0 + \delta_0 - f(\theta)} \tag{56} \]

where \( h(t, \theta) \) is the normalized convergence function. If \( h \) is desired to be a function of time, then \( \gamma \) can be chosen arbitrarily and then \( h \) is numerically obtained, and a function approximation tool of choice can be used to model \( h(t) \).

The other case is where \( h \) is meant to be a periodic function of phase \( h(\theta) \). So the term \( \frac{y_t + \delta_0 - f(\theta)}{y_0 + \delta_0 - f(\theta)} \) should describe a pure periodic behavior multiplied by the exponential decay \( e^{-\gamma t} \). This needs a correct estimation of \( \gamma \). It is difficult, and can also be imprecise, to estimate the shape of \( h(\theta) \) and the convergence factor \( \gamma \) at the same time. The trick here is to calculate \( \gamma \) without knowing the form of \( h(\theta) \). Since \( h(\theta) \) is a non-damped periodic function, its energy in \( 2k\pi \theta + \theta_o, k \in \mathbb{Z} \) phases is equal. This means that if we sample \( h(t) e^{-\gamma t} = \frac{y_{t+0}-f(\theta)}{y_{0}+f(\theta)} \) in \( k\theta + \frac{\theta_o}{2\pi} + t_o \) time stamps, or \( 2k\pi + \theta_o \) phases, then the sampled data fits on a pure exponential decay \( ae^{\gamma t} \) \((t_o \text{ and } \theta_o \text{ are arbitrary offsets})\). Finally, \( \gamma \) can simply be estimated by fitting \( ae^{\gamma t} \) on the newly sampled data, \( e.g. \) by least squares. After \( \gamma \) is estimated, data describing periodic phase-dependent function \( h \) is:

\[ h : \theta_i \mapsto \frac{y_t + \delta_0 - f(\theta_i)}{y_0 + \delta_0 - f(\theta_i)} e^{\frac{\gamma}{2\pi}} \tag{57} \]

and again von Mises bases with locally weighted regression can be used to model this data, like what was done for \( f \).

Figure 11 depicts an example of the procedure to extract the limit cycle and convergence behavior from a given data vector. First the last period of data is used to extract the desired limit cycle describing \( f(\theta) \). Then the data for \( h(t)e^{-\gamma t} \) is formed by equation (56). The value of \( \gamma \) is extracted by fitting an exponential on the \( h(t)e^{-\gamma t} \) data sampled at \( 2k\pi + \theta_o \) phases. Knowing \( \gamma \), we can extract the pure periodic data representing \( h(\theta) \). Finally von Mises bases can be utilized to model the obtained data trajectories for \( f \) and \( h \). Please note that other combinations of limit cycle shape and convergence behavior can also be used, like the ones in figure 10.

### 7.3. Online modulation

Different properties of the proposed morphed oscillators can be modulated online. Frequency modulation can be done by directly changing the \( \theta \) values. This will have an immediate effect on the period of the system. Figure 12-top shows this property. Modulation of amplitude, offset and oscillation midpoint can be done by changing the \( f \) function on-the-fly. If the desired modulated output \( \hat{\Xi}_\theta(\theta) \) is:

\[ \hat{\Xi}_\theta(\theta) = a(\Xi_\theta(\theta) - g) + g + o \tag{58} \]

where \( a \) is the amplitude magnification around midpoint \( g \), and \( o \) is an added offset, then:

\[ \hat{f}(\theta) = af(\theta) + \frac{(1-a)g + o}{\Xi_\theta(\theta)} \tag{59} \]

where \( \hat{f}(\theta) \) is the new scaling function giving the desired modulation. Examples are given in figure 12. The effects of these modulations are not immediate, and act as swapping the limit cycle of the system with a new one, so the system will gradually converge to the modulated limit cycle. All the above holds when \( f \) is replaced with a completely new \( \hat{f} \) which can have a different shape than \( f \), as the bottom plot in figure 12 shows.

If a first order realization is used, then instant switching of \( f \) can result in a momentary discontinuity in the velocity profile. With a second order realization, instant switching of \( f \) can result in a momentary discontinuity in acceleration. In general, with an \( n \)-th order realization, instant switching of \( f \) can result in a momentary discontinuity in \( \dot{\rho}^{(n)} \). If any of these situations are of importance (which depends on the application), and it is not possible to deepen the order of the dynamical system, then a soft switching between \( f \) and \( \hat{f} \) will alleviate the discontinuity issue.

We like to mention that since the oscillators obtained by the introduced methodology are phase oscillators, frequency adaptation rule can easily be applied to them. With reference to [19], a frequency adaptation rule for entrainment [38] with an external periodic input can be simply written as:

\[ \dot{\omega} = -\eta f(t) \sin(\theta) \tag{60} \]

where \( f(t) \) is the external periodic signal and \( \eta \) is the adaptation rate. This means that instead of having a constant \( \omega \) in the phase equation \( \dot{\theta} = \omega \), an adaptive dynamics is applied to \( \dot{\omega} \). Please refer to [19] for details.
Figure 12: Online modulation of the limit cycle behavior. Top) modulation of frequency to $\vartheta = 2$ and then to $\vartheta = 0.5$. Middle) Modulation of amplitude by $a = 0.5$ around midpoint $g = 2.5$, then adding an offset of $o = -2$, and finally going back to the initial limit cycle. Bottom) Swapping the limit cycle with a new one. For all the figures $\gamma = 2$, and except the top figure, the value of $\gamma$ determines the modulation/switching duration.

8. Discussion

We presented a general methodology to morph a chosen phase oscillator, which acts as a base oscillator, to an oscillator with a desired limit cycle shape. The main idea of this methodology is based on a diffeomorphic phase-based scaling map which morphs the dynamics of the base to the desired one. The introduced methodology can be implemented in two forms: 1) the original form where the amplitude of oscillation is scaled by the distance to the limit cycle; and 2) the compensated form where the amplification effect is compensated for. The given methodology creates first order oscillators, and was extended to represent second order or $n$-th order oscillators. This realizes a general and populated family of nonlinear oscillators with any desired order. All the forms and orders exhibit the desired requested limit cycle shape.

Compared to the general approach of using recurrent neural networks to create nonlinear oscillators, using the presented methodology will reduce the design/training complexity. If one desires to obtain a desired limit cycle behavior out of a recurrent neural network, then he/she should employ rather complex training techniques like backpropagation through time [39], and check for local asymptotic stability and unwanted local minima afterwards. There are also easier ways to implement recurrent neural networks exhibiting desired limit cycle behaviors, like the reservoir computing approaches [15]. Nevertheless, all the above approaches need to know about the internal dynamics of the network. However in the case of the methodology introduced in this paper, one only needs to know the limit cycle shape of the base oscillator, and not its transient dynamics, to be able to create the desired nonlinear oscillator. This gives the advantage that the training procedure gets reduced to a static function approximation.

Other than being an interesting mathematical challenge to create nonlinear oscillators with arbitrary limit cycle
shapes, many motion control applications need such oscillators which makes this problem even more interesting. One very good example is the control of locomotion and use of nonlinear oscillators as pattern generators. This is widely known as the problem of designing Central Pattern Generator (CPG) [4] models for locomotion. As Ijspeert mentions in [4], to be able to systematically create CPGs, design of coupled nonlinear oscillator exhibiting desired limit cycles should be tackled. We believe that the approach here is general and systematic and helps to ease the design of CPGs. Moreover, since one can design globally asymptotically stable limit cycle systems with the introduced methodology, the inclusion of feedback signals will not affect the stability properties, and consequently broadens the types of feedback signals that can be included.

The nonlinear oscillators obtained by the introduced methodology are one dimensional phase oscillators, and can get coupled to create a multidimensional system. The whole coupled system has advantages and disadvantages compared to a recurrent neural network which is multidimensional by design. The methodology here makes the creation of a multidimensional system easy by dividing it into low dimensional subsystems. However the radial dynamics of different dimensions are not directly coupled. This means that a perturbation on one dimension’s radial state will not affect the other dimensions, other than when being explicitly coupled. This is different from a recurrent neural network where all the state dynamics are normally coupled. This can be interpreted as an advantage or disadvantage depending on the application. Having the decoupling of the radial dynamics from the phase dynamics, one can control radial dynamics stable point and convergence properties separately and leave the phases for eigenfrequency control or for inter-agents phase synchronization. This motivates applications like CPGs, or more applied examples like model-free tracking in assistance and rehabilitation robotics like [40, 41] when the frequency coupling can be stably separated from radial coupling.

We also introduced the possibility to have an explicitly defined custom convergence behavior for when an amplitude controlled oscillator is used as base. This custom convergence behavior is apart from the choice of the limit cycle shape, and both arbitrary limit cycle shape and convergence can be obtained in one same dynamical system. This gives the possibility to include rhythmic tasks as well as their initiation in one single system. This can be potentially useful in many rhythmic motor control tasks which need initiation, like juggling or locomotion. We have also explained how to learn both the limit cycle shape and the convergence behavior from given data, which makes this tool appropriate to be used for learning rhythmic tasks by imitation.

The stability analysis for the general family obtained from the introduced methodology was given. If one is looking for globally asymptotically stable limit cycle systems, the stability conditions can direct him/her to the choice of the base oscillators he/she has. More importantly, if one is forced to use a specific base (e.g. by implementation constraints), the given stability analysis can be used to know the stability bounds of the resulting system.

We believe that the stability analysis given in section 5 can also be utilized to analyze the stability of a subset of complex systems. To give an example consider the following time-dependent system:

$$\dot{x} = -x \sin(t) + \left(1 - \frac{x^4}{e^{x \cos(t)}} \right) \left( \frac{2x}{e^{x \cos(t)}} + \sin \left( \frac{x}{e^{x \cos(t)}} \right) \right)$$  \hspace{1cm} (61)

It is not trivial to analyze the stability of this time-dependent system and describe its limit cycle properties. However one can reformulate this system as an original morphed oscillator with $f(\theta) = e^{\cos(\theta)}$ and the base:

$$\dot{\theta} = 1$$

$$r_B = (1 - r_B^3)(2r_B + \sin(r_B))$$  \hspace{1cm} (62)

which has one stable fixed point at $r_B = 1$ and an unstable on $r_B = 0$. This base satisfies the conditions in equation (30) for $r_B > 0$, and consequently, utilizing equation (26), the system converges to the time driven limit cycle $e^{cos(t)}$ for $x > 0$. Authors were able to show this because they designed this problem, but with some intuition, the same analogy can be used to analyze the stability conditions of another similar dynamical system. We do not intend to say that this is a standard way to analyze the stability of a limit cycle system, but we believe that it can be helpful in many cases.

We would like to mention that the two layer approach here can, in a very abstract level, be used to model neuroscientific Central Pattern Generators models. As explained in [42], it has been proposed that there are separate

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3 For example Matlab or Mathematica cannot solve this equation.
mechanisms for rhythm generation (i.e. for setting the phase and frequencies) and pattern modulation (i.e. for determining the waveform of the motoneuron output) in mammals. This is similar to having coupled phase dynamics controlling the rhythm of the system, and then having shaping functions describing motoneuron recruitment mechanisms. Of course the approach here is very abstract compared to a detailed neuroscientific model of the Central Pattern Generators, but abstract models can be helpful to explain behaviors at a higher level, like the example in [29].

In the end, we expect the introduced methodology to ease and systematize the design of nonlinear phase oscillators for different applications including motion control. We have already used an example of this tool as the pattern generator of a locomoting quadruped robot [43]. Moreover, as rhythmic DMP is a subset of this family, one can include all the applications of the rhythmic DMP as the applications of this methodology. Additionally, by choosing different bases, researchers can now benefit from a diversity of the different transient nonlinear dynamics that can be useful for rhythmic tasks.

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Dynamically stable control of articulated crowds

Albert Mukovskiy a,*, Jean-Jacques E. Slotine b, Martin A. Giese a

a Section for Computational Sensomotorics, Department of Cognitive Neurology, Hertie Institute for Clinical Brain Research & Center for Integrative Neuroscience, University Clinic Tübingen, Germany
b Nonlinear Systems Laboratory, Department of Mechanical Engineering, MIT, Cambridge, MA, USA

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ABSTRACT

The synthesis of realistic complex body movements in real-time is a difficult problem in computer graphics and in robotics. High realism requires the accurate modeling of the details of the trajectories for a large number of degrees of freedom. At the same time, real-time animation necessitates flexible systems that can adapt and react in an online fashion to changing external constraints. Such behaviors can be modeled with nonlinear dynamical systems in combination with appropriate learning methods. The resulting mathematical models have manageable mathematical complexity, allowing to study and design the dynamics of multi-agent systems. We introduce Contraction Theory as a tool to treat the stability properties of such highly nonlinear systems. For a number of scenarios we derive conditions that guarantee the global stability and minimal convergence rates for the formation of coordinated behaviors of crowds. In this way we suggest a new approach for the analysis and design of stable collective behaviors in crowd simulation.

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1. Introduction

The generation of realistic interactive human movements is a difficult task with high relevance for computer graphics and robotics. Applications such as computer games require online synthesis of such movements, at the same time providing high degrees of realism even for complex body movements. While for the off-line synthesis of human movements, the movements can be recorded off-line and retargeted to the relevant kinematic model, this procedure is not possible for online synthesis. Approaches based on physical or dynamical models (e.g. [1]) have focused on the simulation of scenes with many interacting agents that navigate autonomously and show interesting collective behaviors. Due to the complexity of the underlying mathematical models, such systems are typically designed in a heuristic manner. Opposed to other applications in engineering, the system dynamics of such computer animation systems is usually not analyzed, so that robustness or stability guarantees for the system dynamics cannot be given.

In this paper we present first steps toward the development of more systematic method for the design of the dynamics of interactive crowds. For this purpose, we approximate human movements by relatively simple mathematical models, combining dynamical models with appropriate learning methods. In addition, we introduce Contraction Theory [2] as a new tool for the stability design of complex nonlinear dynamical systems. We demonstrate how this approach can be applied for the stability analysis of groups of autonomously navigating characters with full body articulation. The current research extends our previous work [3] by the development of stability analysis for more complex scenarios, including the control of heading direction and of the consensus behavior of crowds.

The paper is structured as follows: The learning-based dynamical model for complex human movements is briefly sketched in Section 3. Section 4 describes the relevant control dynamics that is required to realize complex interactions between different characters in a crowd. In Section 5 we review basic concepts from Contraction Theory. The major results of our stability analysis and some demonstrations of their application to crowd animation are presented in Section 6, followed by some conclusions.

2. Related work

Dynamical systems have been frequently applied in crowd animation for the simulation of autonomous collective behaviors (e.g. [4,5]). Some of this work was inspired by observations in biology showing that coordinated behavior of large groups of agents, such as flocks of birds, can be modeled as emergent behaviors arising
3. System architecture

The proposed new method for the design of the collective dynamics of interacting crowds is based on a learning-based approach for the modeling of human movements using dynamic movement primitives [20] (cf. Fig. 1). For a relevant class of movements, like gait with different styles or straight vs. curved locomotion, a low-dimensional representation in terms of a small number of basic components is learned using an algorithm for anechoic demixing [21]. We showed elsewhere that this method allows to approximate trajectories by very small number of source signals, outperforming other dimension reduction algorithms, such as ICA or PCA [22,21].

In order to generate the learned source signals online, the stable solutions of a nonlinear dynamical system (dynamic primitive) are mapped onto the form of the source functions. For the synthesis of gait pattern a primitive is naturally modeled by oscillator [23]. The mapping from the phase space of the dynamics, defined by the state variables \( \mathbf{y} = [y, \dot{y}]^T \), onto the values of the source functions \( s_j \) was learned from training data using Support Vector Regression with a Gaussian kernel (see Ref. [20] for details). For the examples presented in this paper each character was modeled by a single Andronov–Hopf oscillator. The generated source signals were then linearly combined with the linear weights \( w_j \) and phase delays \( \tau_j \) in order to generate the joint angle trajectories \( \xi_i(t) \) according to the learned anechoic mixture model that was given by the equation:

\[
\xi_i(t) = \sum_j w_j s_j(t - \tau_j)
\]

(1)

The complete reconstruction of the trajectories requires the addition of the average joint angles \( m_i \), which also were learned from the training data.

For the special case where the dynamic primitives are given by Hopf oscillators, whose limit cycle for appropriate choice of the coordinate system is a circular trajectory in phase space, the phase delays can be absorbed in an instantaneous orthogonal mapping (rotation) \( \mathbf{M}_i \) in the phase plane of the oscillator (Fig. 1b). This allows to derive dynamics for online synthesis without explicit delays, which would greatly complicate the system dynamics. (See Ref. [20] for further details.)

By blending of the mixing weights \( w_j \) and the phase delays \( \tau_j \), intermediate gait styles can be generated. This technique was applied to generate walking along paths with different curvatures, changes in step length, and emotional gait styles. Interactive behavior of multiple characters can be modeled by making the states of the oscillators and the mixing weights dependent on the behavior of the other characters. Such couplings, which are discussed in more detail in the next section, result in a highly nonlinear overall system dynamics. We showed elsewhere that the same architecture can be applied also for the generation of other types of body motion than locomotion [20], while such examples are not discussed in this paper. The walking direction of the characters is also changed by interpolation between straight walking and walking along curved paths to the left or to the right. The parameters used for blending are described in our previous publications [20,24]. Such blending is used to simulate a control of the heading directions in consensus scenarios and for obstacle avoidance during the autonomous reordering of crowds. For the implementation of reactive local obstacle avoidance we used a dynamic navigation model that originally was developed in robotics [25]. See Refs. [24,20] for details concerning the implementation.

4. Control dynamics

Flexible control of the locomotion of articulating agents requires the control of multiple variables, specifying a control dynamics with multiple coupled levels. For the examples discussed in this paper our system included the control of the following variables: (1) phase within the step cycle, (2) step length, (3) gait frequency, and (4) heading direction. The control of step phase was accomplished by coupling of the Andronov–Hopf oscillators [26] that correspond to different agents, resulting in phase synchronization. These oscillators have a stable limit cycle that corresponds to an oscillation with constant amplitude and the (time-dependent) phase \( \phi(t) \). In absence of external couplings the phase increases linearly, i.e., \( \dot{\phi}(t) = \omega \phi(t) \), where \( \omega \) is the stable eigenfrequency of the oscillator. Control of step frequency was accomplished by varying this parameter in a time-dependent manner in dependence of the behavior of the characters in the scene. Step-length and direction were controlled by morphing between gait with different step lengths or path curvatures, blending the parameters of the anechoic mixing model (see above). In this case the controlled variables are the blending coefficients of these mixtures. (See Ref. [20] for details.)

The formulation of the system dynamics in terms of speed control is simplified by the introduction of the positions \( z_i \) for
each individual character along its propagation path (see Fig. 2).

This variable fulfills the differential equation $z_i(t) = \phi_i g(\phi_i)$, where the positive function $g$ determines the instantaneous propagation speed of the character depending on the phase within the gait cycle. This nonlinear function was determined empirically from a kinematic model of a character. By integration of this propagation dynamics one obtains $z_i(t) = G(\phi_i(t) + \phi_0^i) + c_i$, with an initial phase shift $\phi_0^i$ and some constant $c_i$ depending on the initial position and phase of avatar $i$, and the monotonously increasing function $G(\phi) = \int_0^\phi g(\phi) d\phi$, where we assume $G(0) = 0$.

In the following we will analyze four different control rules, whose combination allows to generate quite flexible locomotion behavior of a crowd of characters:

(I) **Control of step frequency**: a simple form of speed control results if the frequency of the oscillators $\phi_i$ is made dependent on the behavior of the other characters. Assuming that $\omega_0$ be the equilibrium frequency of the oscillators without interaction, this can be accomplished by the control dynamics:

$$\dot{\phi}_i(t) = \omega_0 - m_j \sum_{j=1}^N K_{ij} [z_j(t) - z_i(t) - d_{ij}]$$

The constants $d_{ij}$ specify the stable pairwise relative distances in the final ordered form for each pair $(i, j)$ of characters. The elements of the coupling graph’s adjacency matrix $K$ determine whether characters $i$ and $j$ are interacting and thus dynamically coupled. These parameters were set to $K_{ij} = 1$, if the characters were coupled, and they are zero otherwise (with $K_{ii} = 0$). For example, we choose $K_{ij} = 1$, $\forall i \neq j$ for all-to-all coupling, and $K_{ij} = 1$, $\forall j \neq i$ for ring coupling.

With the Laplacian matrix $L^f$ of the coupling graph (that is assumed to be strongly connected [14,27,28]), defined by $L_{ij}^f = -K_{ij}$ for $i \neq j$ and $L_{ii}^f = \sum_{j=1}^N K_{ij}$, and the constants $c_i = -\sum_{j=1}^N K_{ij} d_{ij}$, the last equation system can be re-written in vector form:

$$\dot{\phi} = \omega_0 \mathbf{1} - m_d (L^f G(\phi + \phi^p) + \mathbf{c})$$

(II) **Control of step length**: step length was varied by morphing between gaits with short and long steps. A detailed analysis showed that the influence of step length on propagation speed could be well approximated by simple linear rescaling. If the propagation velocity of character $i$ is $v_i(t) = z_i(t) = \phi_i(t) g(\phi_i(t)) = \omega_0 \phi_i(t) g(\phi_i(t))$ for the normal step size, then the velocity for modified step size could be approximated by $v_i(t) = z_i(t) = (1 + \mu_i) \omega_0 \phi_i(t) g(\phi_i(t))$ with the morphing parameter $\mu_i$. The empirically measured propagation velocity as function of gait phase is shown in Fig. 3(a) for different values of the step length parameter $\mu_i$.

In order to realize speed control by step length the morphing parameter $\mu_i$ was made dependent on the difference between

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**Fig. 1.** (a) Architecture of the system for real-time synthesis of complex human movements. Solutions of dynamical systems (primitives) are mapped onto source signals, that have been derived by anechoic demixing from training data. The solutions of the dynamical systems are mapped by Support Vector Regressions (SVR) onto the source signals. These source signals are then combined using the learned anechoic mixing model to generate joint angle trajectories online, which specify the kinematics of the animated characters. (b) When the dynamic primitives are modeled by nonlinear oscillators the time shifts of the anechoic mixing model can be absorbed in an instantaneous orthogonal transformation $\mathbf{M}_{t_j}$, avoiding a dynamics with explicit delays.

**Fig. 2.** Variables exploited for speed and position control. Every character $i$ is characterized by its position $z_i(t)$, the phase $\phi_i(t)$ and the instantaneous eigenfrequency $\omega_i(t) = \dot{\phi}_i(t)$ of the corresponding Andronov–Hopf oscillator, and a step-size scaling parameter $\mu_i(t)$.

**Fig. 3.** (a) Propagation velocity for different values of the step-length morphing parameter ($\mu_i = [0 \ldots 0.25]$) as function of the gait cycle phase $\phi$. Empirical estimates are well approximated by a linear rescaling of the propagation speed function defined above $v = (1 + \mu)\omega \phi$, for constant $\omega = 1$. (b) The heading direction control depends on the difference between the actual heading direction $\phi^{\text{actual}}$ and the goal direction $\phi^{\text{goal}}$. Movement along parallel lines was modeled by defining ‘sliding goals’ that moved along the lines.
actual and desired position differences $d_{ij}$ between the agents
\[\mu_i = -m_z \sum_{j=1}^{N} K_i^z (z_i(t) - z_j(t) - d_{ij})\]
resulting in the control rule:
\[\dot{z}_i(t) = \omega(t)[\phi(\psi_i(t))] - m_z \sum_{j=1}^{N} K_i^z (z_i(t) - z_j(t) - d_{ij})\]

with the constant coupling strength $m_z > 0$. Here the adjacency matrix $K^z$ of the coupling graph corresponds to the Laplacian matrix $L^z$ (according to the equivalent relationships as specified above). In vector notation the dynamics for the control of speed by step length can be written:
\[\dot{\mathbf{z}} = \omega(t)[\phi(\mathbf{\psi})] - m_z (L^z \mathbf{z} + \mathbf{c})\]  \hspace{1cm} (4)

(III) Control of step phase: by defining separate controls for step length and step frequency the position and step phase of the characters can be varied independently. This makes it possible to simulate arbitrary spatial patterns of characters, at the same time synchronizing their step phases. The additional control of step phase can be accomplished by simple addition of a linear coupling term in Eq. (3):
\[\dot{\mathbf{\phi}} = \omega(t) (1 - m_p L^p G(\mathbf{\phi} + \mathbf{c}) - K^\mathbf{\phi} \mathbf{\phi})\]  \hspace{1cm} (5)
with $k > 0$ and the Laplacian $L^p$. (All sums or differences of angular variables were computed by modulo $2\pi$.)

(IV) Control of heading direction: the control of the heading directions $\psi_i$ of the characters was based on differential equations that specify attractors for goal directions $\psi_i^{\text{goal}}$, which were computed from ‘sliding goals’ that were placed along straight lines at fixed distances in front of the characters (Fig. 3b). The heading dynamics was given by a nonlinear differential equation, independently for every character [20]:
\[\dot{\psi}_i = \omega(t)[-m_p \sin(\psi_i - \psi_i^{\text{goal}}) + g^\mathbf{\psi}(\phi_i(t) + \phi_i^{\text{c}})]\]  \hspace{1cm} (6)
where $\psi_i^{\text{goal}} = \arctan(\Delta z_i^{\text{goal}} / \Delta z_i^{\text{goal}})$, with $\Delta z_i^{\text{goal}}$ specifying the distance to the goal line orthogonal to the propagation direction and $\Delta z_i^{\text{goal}}$ being a constant (Fig. 3b). The first term describes a simple direction controller whose gain is defined by the constant $m_p > 0$. The second term approximates oscillations of heading direction, where $g^\mathbf{\psi}$ is again an empirically determined periodic function. Control is realized by making the morphing coefficients that determine the contributions of left vs. right-curved walking dependent on the change rate $\psi_i$ of the heading direction.

The mathematical results derived in the following sections apply to subsystems derived from the complete system dynamics defined by Eqs. (4), (5) and (6). In addition, simulations will be presented that illustrate the range of behaviors that can be modeled by the full system dynamics.

5. Elements from Contraction Theory

The dynamical systems for the modeling of the behavior of the autonomous characters derived in the last section are essentially nonlinear. In contrast to linear dynamical systems, a major difficulty of the analysis of such nonlinear systems is that stability properties of systems parts usually do not transfer to composite systems. Contraction theory (CT) [2] provides a general method for the analysis of essentially nonlinear systems that permits such a transfer. This makes it suitable for the analysis of complex systems that are composed from components. CT characterizes the system stability by the behavior of the differences between solutions with different initial conditions. If these differences vanish exponentially over time independent from the chosen initial states the system is called contracting. In this case the system is globally asymptotically stable, that is all its solutions converge to a single trajectory independent from the initial state. For a general dynamical system of the form
\[
\dot{x} = f(x, t)
\]
we assume that $x(t)$ is one solution of the system and $\tilde{x}(t) = x(t) + \delta x(t)$ a neighboring one with different initial condition. The function $\delta x(t)$ is also called virtual displacement. With the Jacobian of the system $J(x, t) = \partial f(x, t)/\partial x$ it can be shown [2] that any virtual displacement decays exponentially to zero over time if the symmetric part of the Jacobian $J = (J + J^T)/2$ is uniformly negative definite, denoted as $J < 0$. If, i.e. has negative eigenvalues for all relevant state vectors $x$. In this case, it can be shown that the norm of the virtual displacement decays at least exponentially to zero for $t \to \infty$. If the virtual displacement is small enough, then
\[
\frac{d}{dt}||\delta x(t)||^2 = 2\delta x^T(t)J(x, t)\delta x
\]
implies through $\frac{d}{dt}||\delta x(t)||^2 \leq ||\delta x(t)||^2 e^{-\lambda_{\text{max}}||J(x, t)||t}$. The decay of the virtual displacement occurs thus with a convergence rate (inverse timescale) that is bounded from below by the contraction rate $\rho_c = -\sup_{x} \lambda_{\text{max}}(J(x, t))$, where $\lambda_{\text{max}}(.)$ signifies the largest eigenvalue. This has the consequence that all trajectories converge to a single solution exponentially in time [2].

Contraction analysis can be applied to hierarchically coupled systems that are given by the dynamics
\[
\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1) \\ f_2(x_1, x_2) \end{pmatrix}
\]
where the first subsystem is not influenced by the state of the second. The corresponding Jacobian $F = \begin{pmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{pmatrix}$ implies for the dynamics of the virtual displacements:
\[
\frac{d}{dt} \begin{pmatrix} \delta x_1 \\ \delta x_2 \end{pmatrix} = \begin{pmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \end{pmatrix}
\]
If $F_{21}$ is bounded, then the exponential convergence of the first subsystem, (following from $|F_{21}| < 0$), implies thus convergence of the whole system, if in addition $F_{22} < 0$. This follows from the fact the term $F_{21}\delta x_1$ is just an exponentially decaying disturbance for the second subsystem. (See Ref. [2] for details of proof.)

In practical applications many systems are not contracting with respect to all dimensions of the state space, but rather show convergence only with respect to a subset of dimensions. This behavior can be mathematically characterized by partial contraction [18,28]. The underlying idea is the construction of an auxiliary system that is contracting with respect to a subset of dimensions (or submanifold) in state space. The major result is the following [18]:

**Theorem 1.** (Partial contraction) Consider a nonlinear system of the form $\dot{x} = f(x, x, t)$ and the auxiliary system $\dot{y} = f(y, x, t)$. If the auxiliary system is contracting with respect to $y$ uniformly for all relevant $x$ then the original system is called partially contracting. This implies that if a particular solution of the auxiliary system verifies a specific smooth property then all trajectories of the original system also verify this property with exponential convergence.

A ‘smooth property’ is a property of the solution that depends smoothly on space and time (assuming the relevant derivatives or partial derivatives exist and are continuous), such as convergence against a particular solution or a properly defined distance to submanifold in phase space [18]).

It thus is sufficient to show that the auxiliary system is contracting to prove convergence to a subspace. If the original system has a flow-invariant linear subspace $M$, which is defined by the property that trajectories starting in this space always remain in it (Yi: $f(M, t) \subseteq M$), and assuming that the matrix $V$ is an orthonormal projection onto $M^\perp$, then a sufficient condition for global exponential convergence to $M$ is given by [27,28]:

$$\dot{V} = \left( \frac{\partial f}{\partial x} \right) V < 0,$$

(9)

where $A < 0$ again indicates that the matrix $A$ is negative definite.

Finally, we introduce here a theorem that provides sufficient conditions for the synchronization of a network that is composed from $N$ identical dynamical systems that communicate through a common medium or channel with state variable $\chi$. The relevant dynamics is given by

$$\dot{x} = f(x, \chi, t),
\dot{\chi} = g(\chi, \Psi(x), t),$$

(10)

$x$ containing the state variables of the individual systems and all components of $f$ having the same form $f$. Exploiting the last Partial contraction theorem the following result can be derived [29]:

**Theorem 2.** (Quorum sensing) If the reduced order virtual system $\dot{y} = f(y, \chi, t)$ is contracting for all relevant $\chi$ then all solutions of the original system converge exponentially against a single trajectory, i.e. $|x_i(t) - x_j(t)| \to 0$ as $t \to + \infty$.

6. Results: stability conditions for crowd control

In the following we derive stability conditions for the formation of coordinated behavior of groups, providing contraction bounds for four scenarios corresponding to control problems with increasing levels of complexity. Corresponding crowd behaviors are illustrated by demo movies that are provided as supplements for the manuscript.

(1) Control of step phase without position control: this simple control rule permits to simulate step synchronization, as in the case of a group of soldiers [28, [Demo 1]. The dynamics for this case is given by Eq. (5) with $m_q = 0$ (omitting the position control term). For $N$ identical dynamical systems with symmetric identical coupling gains $K_{ij} = K_{ji} = k$ the dynamics can be written

$$\dot{x}_i = f(x_i) + k \sum_{j \in N_i} (x_j - x_i), \quad \forall i = 1, \ldots, N$$

(11)

where $N_i$ defines the index set specifying the neighborhood in the coupling graph, i.e. the other characters that are directly interacting with character $i$. The system can be rewritten compactly: $\dot{x} = f(x, t) - kLx$ with the concatenated phase variable $x = [x_1^T, \ldots, x_N^T]^T$. The matrix $L = L_G \otimes L_p$ is derived from the Laplacian matrix of the coupling graph $L_G$, where $p$ is the dimensionality of the individual sub-systems ($L_p$ is the identity matrix of dimension $p$, and $\otimes$ signifies the Kronecker product). The Jacobian of this system is given by $J(x,t) = Df(x,t) - kL$, where the block-diagonal matrix $Df(x,t)$ contains the Jacobians of the uncoupled components $\frac{df}{dx}(x_i, t)$. The dynamics has a flow-invariant linear subspace $M$ that contains the particular solution $x_1 = \cdots = x_N$. For this solution all state variables $x_i$ are identical and thus in synchrony. In this case, the coupling term in Eq. (11) vanishes, so that the form of the solution is identical to the one of an uncoupled system $\dot{x}_i = f(x_i)$. If $V$ is a projection matrix onto the invariant subspace $M^\perp$, then by Eq. (9) the sufficient condition for convergence toward $M$ is given by $V(D(x,t) - kL) = V < 0$. This implies $\lambda_{x_i}(V(kL) = k \lambda_{x_i}^L > \sup_{x_i} \lambda_{x_i}^L (D_i)$, with $\lambda_{x_i}^L$ being the smallest non-zero eigenvalue of $M^{L_{ij}}$, of the Laplacian $L_i$. For strongly connected coupled graphs all the nonzero eigenvalues of $L_i$ are real positive, due to the Gershgorin’s disc theorem [30]. The sufficient condition for global stability of the overall system is given by $k > \sup_{x_i} \lambda_{x_i}^L (D_i)$. This implies the minimum convergence rate: $\rho_{x} = - \sup_{x_i} \lambda_{x_i}^L (D_i)$.

(2) Speed control by variation of step frequency:

the dynamics of this scenario is given by Eqs. (3) and (4) for $m_q = 0$. Assuming arbitrary initial distances and phase offsets of different propagating characters, implying by $G(\phi_i) = c_i$ that $c_i \neq c_j$, for $i \neq j$, we redefine $d_j = d_j - (c_i - c_j)$ in Eq. (2), and accordingly $c$ in Eq. (3). Furthermore, we assume for this analysis a scenario where the characters follow one leading character whose dynamics does not receive input from the others. In this case all phase trajectories converge to a single unique trajectory only if $c_i = c_j$ for all $i$, $j$, as consequence of the strict correspondence between gait phase and position that is given by Eq. (3). In all other cases the trajectories of the followers converge to one-dimensional, but distinct attractors that are uniquely defined by $c_i$. These attractors correspond to a behavior where the followers’ positions oscillate around the position of the leader. The partial contraction of the dynamics with $c = 0$ guarantees that the resulting attractor area is bounded in phase space (cf. Ch. 3.7.7 in Ref. [22]).

For the analysis of contraction properties we regard an auxiliary system obtained from Eq. (3) by keeping only the terms that depend on $\phi_i$: $\dot{\phi}_i = -m_iL_i^2G(\phi_i + \phi_i)^0$. According to Theorem 1 the symmetrized Jacobian of this system projected onto the orthogonal complement of the flow-invariant linear subspace $\phi_i^0 + \phi_i^0 = \cdots = \phi_i^0 + \phi_i^0$ determines whether this system is partially contracting. By virtue of a linear change of variables, the study of the contraction properties of this system is equivalent to study the contraction properties of the dynamical system $\dot{\phi}_i = -m_iL_i^2G(\phi_i)$ on trajectories converging toward the flow-invariant manifold $\phi_i^0 = \cdots = \phi_i^0$.

The sufficient conditions for (exponential) partial contraction toward flow-invariant subspace are, (see Eq. (9)):

$$V_i(\phi_i)^0 = -m_iV_i(\phi_i)^0 < 0, \text{introducing } B(\phi) = L_i^2D_i + D_i(L_i^2)^T \text{ and }$$

signifying the projection matrix onto the orthogonal complement of the flow-invariant linear subspace. For diffusive coupling with symmetric Laplacian the linear flow-invariant manifold $\phi_i^0 = \cdots = \phi_i^0$ is also the null-space of the Laplacian. In this case, the eigenvectors of the Laplacian that correspond to nonzero eigenvalues can be used to construct the projection matrix $V_i$. For example, in the case of $N$ characters with symmetrical all-to-all coupling with $L_i = M_i - 11^T \geq 0 \text{ we obtain } V_i(L_iD_i + D_i(L_i^2)^T)^0 = \lambda_iM_iV_i^0 > 0 \text{ for } D_i > 0$.

If this case the contraction rate is given by $\rho_{\phi_i} = m_i\min_{\phi} \lambda_i(\phi_i)^0$, with $\lambda_i = \lambda_i^\perp$ for $\phi_i = 0$.

For general symmetric couplings with positive links with equal coupling strength $m_q > 0$ a sufficient contraction condition is:

$$\lambda_{\min}(M_i^\perp) / \lambda_{\max}(M_i^\perp) > \max_{\phi_i} (g(\phi_i) - \min_{\phi_i} (g(\phi_i))) / \min_{\phi_i} (g(\phi_i)),$$

with $\min_{\phi_i} (g(\phi_i)) = 1/T \int_0^T g(\phi_i)d\phi$. This condition was derived from the fact that for symmetric (positive) matrices $M_1$ and $M_2$ for $(M_1 - M_2) > 0$ it is sufficient to satisfy $\lambda_{\min}(M_1) > \lambda_{\max}(M_2)$.

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1 http://www.uni-tuebingen.de/uni/kmv/artl/avi/cj2012/video0.avi

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sufficient condition permits to constrain the admissible coupling topologies dependent on the $g(\phi)$. Alternatively, it is possible to introduce low-pass filtering in the control dynamics to increase the smoothness of the function $g(\phi)$, see Ref. [3].

These stability bounds are illustrated by [Demo2] that shows convergent behavior of the characters when the contraction condition $m_3 > 0$, $L^3 > 0$ is satisfied for all-to-all coupling. [Demo3] shows the divergent behavior of a group when this condition is violated when $m_3 < 0$. In these and the next demonstrations the actual values of interaction parameters $m_4$, $m_2$, $m_3 > 0$ (in cases, when they additionally satisfy the sufficient contraction conditions) were obtained by matching the corresponding convergence rates to those of the real human behavior in crowds [31].

(3) Step size control combined with control of step phase: the dynamics is given by Eqs. (4) and (5) with $m_2 = 0$. This dynamics defines a hierarchically coupled nonlinear system (in type of Eq. 8). While the dynamics would be difficult to analyze with classical methods, the dynamics for $\mathbf{z}(t)$ that is given by Eq. (4) is partially contracting in the case of all-to-all coupling for any bounded external input $\phi(t)$ if $m_2 > 0$, $L^2 > 0$, and $\omega(t) > 0$. These sufficient contraction conditions can be derived from the requirement of the positive-definiteness of the symmetrized Jacobian applying similar techniques as above. The Jacobian of this subsystem is $J(\phi, \omega) = -m_2 D^2(\phi, \omega) L^2$, with the diagonal matrix $D^2(\phi, \omega) = \omega g(\phi + \phi^0)$ that is positive definite since $g(\phi) > 0$ and $\omega > 0$. This subsystem is (exponentially) contracting and its relaxation rate is determined by $\rho_2 = m_2 \min(g(\phi)) \lambda^+_{L^2}$ (in the case of all-to-all coupling) for any input from the dynamics of $\phi(t)$, cf. Eq. (5). The last dynamics is contracting when $L^3 > 0$ and its relaxation rate is $\rho_3 = \lambda^+_{L^3}$, where $\lambda^+_{L^3}$ is the smallest non-zero eigenvalue of $L^3$. The effective relaxation time of the overall dynamics is thus determined by the minimum of the contraction rates $\rho_2$ and $\rho_3$ (Fig. 4).

Demonstrations of this control dynamics satisfying the contraction conditions are shown in [Demo4], without control of step phase, and in [Demo5], with control of step phase.

(4) Advanced scenarios: a simulation of a system with stable dynamics including both types of speed control (via step size and step frequency) and step phase control is shown in [Demo6], and a larger crowd with 16 avatars simulated using the opensource animation engine Horde3d [32], is shown in [Demo7]. In this simulation an additional dynamics for obstacle avoidance and the control of heading direction was activated during the unsorting of the formation of avatars. Then this navigation dynamics was deactivated, and speed and position control according to the discussed principles result in the final coordinated behavior of the crowd. (See [Demo8].) Finally, [Demo9] shows the divergence of the dynamics for $m_4 < 0$, violating the contraction condition for the step phase dynamics. The two simulations shown in [Demo10] and [Demo11] illustrate the convergence for a crowd with 49 avatars for two different values of the strength of the distance-to-step size coupling, the parameters of step phase coupling remaining constant. The development of stability bounds and estimates of relaxation times for more complicated scenarios is the goal of ongoing work.

(5) Control of heading direction: for the control of heading direction in presence of couplings that affect the step phases, the contraction conditions can be derived exploring the result on hierarchically coupled systems discussed in Section 5. For the analysis of the stability of the dynamics defined by Eq. (6) it is thus sufficient to analyze the contraction properties of the dynamics for the heading direction $\psi$, treating the additional term $\alpha(t)g(\phi(t))$ as an external input to the $\psi$ subsystem.

Assuming a constant goal direction, it was shown in Ref. [2] (Ch. 3.9) that the uncoupled dynamics for one character, given by $\dot{\psi} = -\omega(t) m_4 \sin(\psi - \psi_{\text{goal}})$ is contracting in the intervals $\mid \psi_{\text{goal}} + 2\pi n, \psi_{\text{goal}} + 2\pi n \mid$, $n \in \mathbb{Z}$ for constant $m_4 > 0$. (If $\phi(t)$ is a smooth strictly increasing function of $t$ with the substitution $\phi(t) = \psi(t)$ (and $\omega(t) = d\phi/dt$) the last differential equation can be rewritten then: $d\psi/dt = m_4 \sin(\psi(t) - \psi_{\text{goal}})$).

Another possibility is to realize direction control is to feed back the circular mean average direction of all characters as joint control parameter $\chi = \angle(1/N \sum \psi_i \sqrt{N})$. In this case the dynamics is given by

$$\dot{\psi}_i = \alpha(t) \sin(\chi - \psi_i) + g(\phi(t)), \forall i \in [1 \ldots N],$$

which is suitable for the application of Theorem 2. This implies that the overall dynamics is contracting if the dynamics $\dot{\psi}_i = \alpha(t) \sin(\chi(t) - \psi_i)$ is contracting for any $\chi(t)$. The same Theorem guarantees contraction, when the consensus variable $\chi$ is estimated by a low-pass filter (with time-constant $\alpha > 0$): $\chi = \alpha \chi + (1 - \alpha) \angle(1/N \sum exp(\hat{\psi}_i \sqrt{N})).$ The simulation shown in [Demo12] illustrates the consensus scenario defined by Eq. (12), (without a synchronization of gait cycles).

Conclusions

The analysis and design of the dynamic properties of the formation of ordered patterns in crowds so far has been only rarely treated in computer animation, and treatments in control theory typically assume highly simplified agent models. To our knowledge, this paper presents the first systematic treatment of the dynamics of order formation in crowds using more complex agent models that include articulation of the characters during locomotion. Combining a set of specific approximations of the system dynamics with Contraction Theory as mathematical framework for the systematic treatment stability properties of complex nonlinear systems, we presented a number of examples for the derivation of stability.
bounds for nontrivial scenarios of coordinated crowd behavior during locomotion. We think that the shown examples demonstrate the feasibility of the applied approach and make it plausible that it can be extended for even more complex scenarios. Necessarily, this first exploratory study is highly incomplete and the spectrum of analyzed behaviors of crowds is still very limited. Future work will have to add other dynamical primitives to the model architecture, including ones suitable for the realization of other behaviors than locomotion. The integration of such additional components will necessitate the development of new approximations and applications of additional methods from nonlinear control theory in order to derive the relevant contraction bounds. This defines the research agenda for our future work.

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